LESSONS

IN

APPLIED MECHANICS

LESSONS

IN

APPLIED MECHANICS

BY

JAMES H. COTTERILL, F.R.S.

AND

JOHN HENRY SLADE, R.N.

London

MACMILLAN AND CO.

AND NEW YORK

1891

PREFACE

THE present volume consists in great measure of selections from the matter contained in a larger treatise[1] on the same subject, of which it may therefore in a certain sense be described as an abridged edition. The portions selected have, however, been completely rearranged and rewritten in fuller detail with much additional illustration, and the whole consequently forms a virtually new book, which, being smaller in compass and more elementary in character, may, it is hoped, serve as an introduction to the larger work, and meet the wants of junior students of engineering and others commencing the study of the subject.

The want of a book of this class having been felt for use at the engineer school at Keyham and elsewhere, the writer considered with some care the most suitable plan of arrangement, the method of treatment, and the

[1] *Applied Mechanics*, by James H. Cotterill, F.R.S., Professor of Applied Mechanics in the Royal Naval College, Greenwich.

additional matter required. Subsequently, his time and attention being fully occupied with other work, the task of preparing the present treatise was entrusted to his friend and (late) assistant, Mr. J. H. Slade, who undertook to provide both text and examples, in accordance with the original plan. The examples, over 250 in number, have with few exceptions been constructed especially for this book, and it is hoped will be found to add considerably to its utility.

GREENWICH, *November* 1890.

CONTENTS

PRELIMINARY CHAPTER

Mensuration and Curve Construction

Part I

The Principle of Work

CHAPTER I

Motion—Sliding, Turning, and Screw Pairs

CHAPTER II

Efforts and Resistances—Friction

CHAPTER III

Work and Energy

CHAPTER IV

Oblique and Variable Forces—Indicator Diagrams

CHAPTER IX

THE DIRECT ACTOR (*Continued*)—FORCES

CHAPTER X

UNBALANCED FORCES

CHAPTER XI

THE DIRECT ACTOR—FLUCTUATION OF ENERGY AND SPEED

CHAPTER XII

DYNAMOMETERS—BRAKES—GOVERNORS

Part II

STRENGTH OF MATERIALS AND STRUCTURES

CHAPTER XIII

STRESS AND STRAIN—TENSION AND COMPRESSION

CHAPTER XIV

TRANSVERSE LOADS—BENDING AND SHEARING

CHAPTER XV

Distributed Loads—Principle of Superposition

CHAPTER XVI

Openwork Beams—Warren and N Girders

CHAPTER XVII

Lattice Girder—Travelling Load—Counterbracing

CHAPTER XVIII

Bending of Solid Beams

CHAPTER XIX

Bending (*Continued*)

CHAPTER XX

Shearing and Torsion

CHAPTER XXI

Experimental Results—Elasticity—Strength—Resistance to Impact

CHAPTER XXII

Framework Structures—Simple Triangular Frames

Part III

Hydraulics

CHAPTER XXIII

Elementary Principles of Hydraulics

CHAPTER XXIV

Hydraulic Machines

PRELIMINARY CHAPTER

MENSURATION AND CURVE CONSTRUCTION

FOR the solution of many questions in Applied Mechanics, it is necessary that the student should be able to find the periphery or area of a plane figure of given dimensions, or the surface or volume of a solid body.

Also the construction of certain curves is often necessary; and a knowledge of a few simple properties of these curves enables us to solve questions with greater ease and rapidity than could otherwise be attained. Hence we commence with the present chapter.

Most of the forms we meet with are of simple character, or else can be readily resolved into such simple ones; we consider then these simple forms in order.

The Parallelogram.

RULE.—*The area of a parallelogram is obtained by multiplying any side by the perpendicular distance between it and the opposite side.*

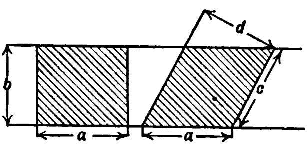


Fig. 1.

Thus in the figure we have two parallelograms, but the area of each is $a \times b$ square inches or feet, according as a and b are inches or feet.

Or, if it be more convenient, the area of the right-hand figure is $c \times d$.

The Triangle.

RULE.—*The area of a triangle is obtained by multiplying any side by the perpendicular distance of the opposite corner from it, and dividing the result by 2.*

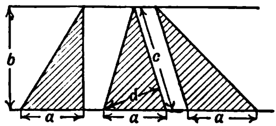


Fig. 2.

The three triangles shown have then each the same area, viz. $\frac{a \times b}{2}$.

Or we can, as in the centre triangle, express it as $\frac{c \times d}{2}$.

There are other rules which may be used, generally involving the trigonometrical functions of the angles, depending on what data are given. But the engineer should generally, in such cases, use the given dimensions to construct the figure to scale; and then measure the necessary dimensions required for the preceding rule.

EXAMPLE.—Find the area of a triangle, the sides of which are 7, 8, and 9 inches respectively.

If the triangle be constructed to scale, on the 8-inch side as base, and the perpendicular height is then measured, it will be found to be 6.7 inches.

$$\therefore \text{ Area} = \frac{8 \times 6.7}{2} \text{ sq. in.,}$$
$$= 26.8 \text{ sq. in.}$$

The Trapezium.—This figure has one pair of sides parallel, but not the other.

RULE.—*The area of a trapezium is obtained by multiplying half the sum of the parallel sides by the perpendicular distance between them.*

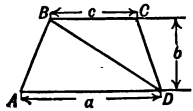


Fig. 3.

Hence area of figure $= \frac{a+c}{2} \times b$.

We can see how this rule is obtained if we divide the figures into two triangles by the line BD.

We then have

$$\text{Area ABCD} = \text{ABC} + \text{BCD},$$
$$= \frac{ab}{2} + \frac{cb}{2},$$
$$= \frac{a+c}{2} \cdot b.$$

Any Figure bounded by Straight Lines.—The method just given can evidently be extended, and we thus obtain—

RULE.—*To obtain the area of any rectilinear figure. Divide the figure into triangles, or parallelograms, and find the area of each separately. Finally add all the results together.*

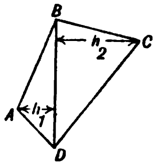


Fig. 4.

We will apply this rule to the case of a trapezoid or quadrilateral; this is a four-sided figure having no parallel sides. Quadrilateral, however, includes all cases, parallel or not.

ABCD is the given quadrilateral. Divide into triangles as shown. Then

$$\text{Area ABCD} = \text{ABD} + \text{BCD},$$
$$= \tfrac{1}{2}.\text{BD} \times h_1 + \tfrac{1}{2}\text{BD} \times h_2,$$
$$= \tfrac{1}{2}\text{BD}\,(h_1 + h_2).$$

More complex cases are solved in an exactly similar manner, the only difficulty being in deciding on the manner of dividing the area. For this no rules can be given; practice alone will enable the student to decide on the quickest method for any given case.

We come now to areas enclosed by curved lines, of which the simplest is

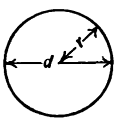


Fig. 5.

The Circle.

RULE.—*The area of a circle is obtained by squaring the diameter, and multiplying the result by* .7854. *Or squaring the radius and multiplying by* 3.1416.

The number 3.1416, of which .7854 is one-fourth, is denoted by π; and hence in the figure

$$\text{Area} = \frac{\pi}{4}d^2 \text{ or } \pi r^2.$$

[For most practical purposes π may be taken as $\frac{22}{7}$.]

In rectilinear figures the periphery or circumference can be directly measured; but this cannot be done in curved figures, hence we require a rule.

RULE.—*The circumference of a circle is obtained by multiplying the diameter by π.*

Hence in figure 5

$$\text{Circumference} = \pi d \text{ or } 2\pi r.$$

The Ring.—This may represent the cross-section of a hollow shaft, or column.

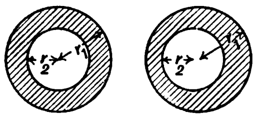


Fig. 6.

From the preceding we obtain

$$\begin{aligned} \text{Area of ring} &= \pi (r_1^2 - r_2^2) && (1), \\ &= \pi (r_1 - r_2)(r_1 + r_2) && (2), \\ &= \text{mean circumference} \times \text{thickness}. \end{aligned}$$

If a table of squares be available, use (1), otherwise use (2).

It is immaterial whether the hole be concentric with the outside or not. The equations are the same. But the result in words would not be true unless by "thickness" *mean* thickness be understood.

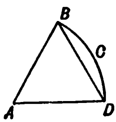


Fig. 7.

Sector of a Circle.—ABCD is the sector. The angle BAD is its angle.

RULE.—*The area of a sector is obtained by multiplying the area of the whole circle by the ratio of the angle of the sector to four right angles.*

Segment of a circle.—BCD (Fig. 7) is the segment. Its area is best found by subtracting the triangle ABD from the sector ABCD.

The Ellipse.—This is the figure produced when a circular cylinder is cut across in any direction not perpendicular to its axis. It is not of great importance to us, so we shall not examine its properties. There are several methods of constructing an ellipse, of which we will give one, which is simple, and assists the student to remember the value of the area.

To construct an ellipse with given axes.

AB and CD are the given axes, O the centre. The lengths of the semi-axes are a and b respectively, a being the major or greater.

On AB, with centre O, describe a circle. Divide AB into a number of parts, and draw ordinates of the circle as shown. Reduce each ordinate in the proportion of b to a. Then through the tops of the reduced ordinates draw a curve, which is the required ellipse.

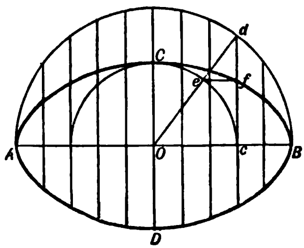


Fig. 8.

The method of reduction is shown for the ordinate cd. The inner circle with radius b is described, d is joined to O cutting the inner circle in e, and ef is drawn parallel to AB, cutting cd in f, then f is one of the points.

For $cf : cd :: b : a$ (similar triangles).

Only one quarter need be drawn in this manner, the others are quite symmetrical.

Now each ordinate of the circle is diminished in the ratio b/a, hence evidently so is the whole area.

$$\therefore \text{ Area of ellipse} = \pi a^2 \times \frac{b}{a} = \pi ab.$$

Hence one rule will suffice both for ellipse and circle as follows.

RULE.—*The area of an ellipse is $\pi/4$ times that of the circumscribing rectangle.* The circle is then a particular case, where $b=a$, and the rectangle becomes a square.

The Parabola.—This curve is the most important we have to deal with, its construction enabling us to solve graphically many problems.

To construct a parabola of given height on a given base. First case.

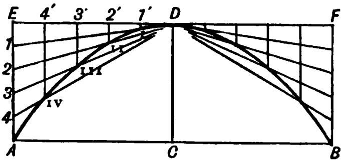


Fig. 9.

Let AB be the given base; and D the *apex* be directly over C, the middle point of AB. The parabola will then be symmetrical, CD being the given height.

Complete the rectangle AEFB.

Divide DE into any number of equal parts (in the figure we take five), and EA into the same number.

Number the divisions 1′, 2′, etc., from D, and 1, 2, etc., from E as shown.

Join D to 1, 2, 3, and 4.

Through 1′ draw a vertical cutting D1 in I; through 2′ a vertical cutting D2 in II; and so on, obtaining the points I, II, III, IV. Then the curve is drawn through A, the above four points, and D.

Similarly for the other side.

DC is called the *axis*, D the *apex;* and the nature of the curve is such that the distance of any point in it from DE varies as the square of its distance from DC. It will be seen that the construction fulfils this condition.

For example, IV is twice as far from DC as II is, but is 4, *i.e.* 2^2, times as far from DE.

The parabola may also be required to have its greatest height at some point not directly over C.

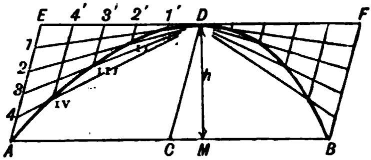


Fig. 10.

AB is again the base, and D the given highest point, at a height *h* say over M.

Join CD, and complete the parallelogram AF.

The construction now proceeds exactly as in the preceding case, the ordinates being drawn parallel to CD.

The two sides are now unsymmetrical, CD is not *the* axis, nor is D the apex. CD is, however, *an* axis.

The curve has an important property, which we shall require to use hereafter, viz. *The tangent at any point bisects the abscissa of that point.*

Thus take for example IV, in either figure, then the tangent at IV passes through 2′, *i.e.* bisects D4′.

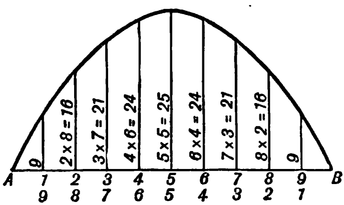


Fig. 11.

Another method of constructing the curve, showing yet another property, is illustrated in Fig. 11.

This method is of use in cases where the actual height in inches is not fixed. AB is divided into ten parts, and they are numbered from each end as shown. Then at each point set up an ordinate representing the product of the two numbers at the point; either vertically, as in the figure, or all parallel in any direction.

The curve then passes through the tops of the ordinates.

The property which leads to this method of construction is, in words—

The ordinate is proportional to the rectangle contained by the segments into which it divides the base.

For the area we have—

RULE.—*The area of a segment of a parabola is* $\frac{2}{3}$ *of that of the circumscribing parallelogram;* i.e. $\frac{2}{3}$ *of the product of base and perpendicular height.*

The circumscribing parallelogram means the parallelogram AEFB in Figs. 9 or 10.

The Hyperbola.—This curve together with the two preceding constitute the conic sections, which are treated in Analytical Geometry. For our purposes we select only those properties which are directly useful; and we define the curve by those properties, and not as it is defined in geometry.

For our purposes then we define the curve thus:—

The hyperbola is the curve traced by a point which moves so that the product of its distance from two rectangular axes is constant.

This is strictly a special case, and is the *rectangular* hyperbola. It is, however, the only one we concern ourselves with.

Construction of the hyperbola.

Let OX, OY be the axes, and A one known position of the moving point. Through A draw lines AN and AM, parallel to OX and OY respectively. Produce AN as far as necessary. (The curve has no limits.)

Mark off along MX distances M1, 12, etc., equal or

not, so far as the curve is required to extend, and draw the ordinates 11, 22, etc. Join now the *top* points 1, 2, etc., to O, cutting MA in 1′, 2′, etc.

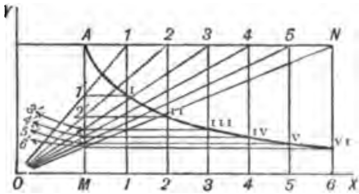


Fig. 12.

Through 1′ draw 1′I parallel to OX, cutting 11 in I; through 2′, 2′II cutting 22 in II; and so on for III, IV, V, etc.

Then I, II, etc., are points on the required curve, and we draw it through them.

By Euclid we easily prove that the rectangles, OI, OII, etc., are equal to OA.

Area.—The particular area required is that bounded by the curve, two ordinates, and OX; such as ABCD in Fig. 13.

For this we have

$$\text{Area ABCD} = \text{AD} \times \text{OD} \times \log_e \frac{\text{OC}}{\text{OD}},$$

$$\text{or equally BC} \times \text{OC} \times \log_e \frac{\text{OC}}{\text{OD}}.$$

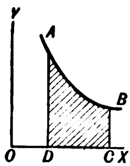


Fig. 13.

The log. is the Napierian or natural log., or from its present property the hyperbolic log., and is 2.3026 times the ordinary tabular log. The number 2.3026 being known as the *modulus.*

Any Curve.

To find the area between the irregular curve CD, the ordinates AC and BD, and the base AB; we proceed as follows:—

Divide AB into an *even* number of *small* parts, and draw ordinates.

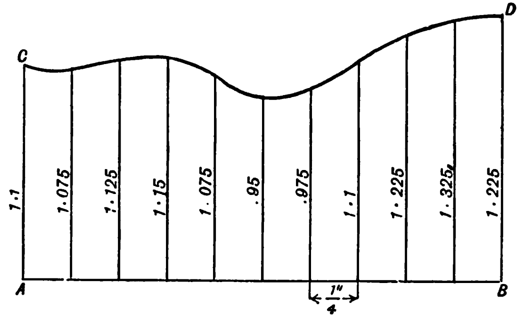


Fig. 14.

Let x = length of each small part.
Add together separately

1st. The two extreme ordinates = A say,
2d. The *remaining odd* ordinates = B say,
3d. The *even* ordinates = C say,

then

$$\text{Area ABCD} = (A + 2B + 4C)\frac{x}{3},$$

or in words—

The sum of the extremes, four times the even, and twice the remaining odd; all multiplied by one-third of the common interval.

For example in Fig. 14 the lengths are given in inches, the interval being $\frac{1}{4}''$. Then

Extremes $1.1 + 1.225 = 2.325$.
Remaining odd $1.125 + 1.075 + .975 + 1.225 = 4.4$.
Evens $1.075 + 1.15 + .95 + 1.1 + 1.325 = 5.6$.

$$\therefore \text{ Area} = \frac{(2.325 + 2 \times 4.4 + 4 \times 5.6)\frac{1}{4}}{3} \text{ sq. ins.} = 2.8 \text{ sq. ins.}$$

Peripheries.—These are not of importance; and can-

not in most cases be accurately found, hence we omit them except for the circle, which is given on page 4.

We come next to solid figures. These we treat briefly.

Square-edged Plate.

RULE.—*The volume of a piece of plate of any shape is equal to the product of the area of its face and its thickness.*

Volumes are required chiefly for the determination of weights. To determine this we must know the weights of unit volumes of the particular materials, which will for the principal ones be found in a table in chap. xxi.

The Sphere.

RULE.—*The volume of a sphere of radius r is* $\frac{4}{3}\pi r^3$.

RULE.—*The surface of a sphere of radius r is* $4\pi r^2$.

The Cylinder.

Fig. 15 shows two views of a cylinder. Side view and end view or cross-section. This is a right circular cylinder, the two ends being perpendicular to the axis.

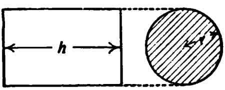


Fig. 15.

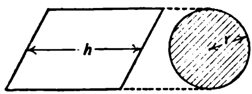


Fig. 16.

Fig. 16 shows a circular cylinder, the two ends not being perpendicular to the axis but parallel to each other. The two cross-sections are the same. Then for each case we have—

RULE.—*The volume of a cylinder is obtained by multiplying the area of its cross-section by its length measured along the axis.*

Thus in each case

$$\text{Volume} = \pi r^2 h.$$

RULE.—*The surface of a cylinder is obtained by multiplying the periphery of its cross-section by its length along the axis.*

In each case then—

$$\text{Surface} = 2\pi r h.$$

If the cylinder be hollow and thin, and cut through along *ab*, then it can be flattened into a plate; the area of which is evidently as above.

The Parallelopipedon.—This is a solid, whose cross-section is not circular, but is the same at all points of its length; as for example a wrought or cast-iron beam (Fig. 17).

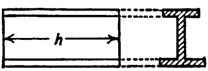


Fig. 17.

The rules for the cylinder apply exactly.

Lastly, though of minor importance, we will take—

The Cone.

RULE.—*The volume of a cone is $\frac{1}{3}$ that of a cylinder on the same base, and of the same height.*

$$\therefore \text{ Volume} = \tfrac{1}{3}\pi r^2 h.$$

The curved surface can be found by supposing the cone hollow, cutting it down a line as *ab*, and flattening it out as shown in Fig. 19, forming a section of a large circle of radius *l*.

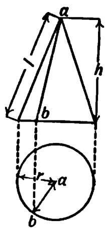


Fig. 18.

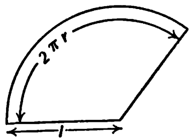


Fig. 19.

Then we have

$$\text{Area} = \text{area of the whole circle} \times \frac{2\pi r}{\text{whole circumference}},$$

$$= \pi l^2 \times \frac{2\pi r}{2\pi l},$$

$$= \pi l r,$$

or in words—

The curved surface is obtained by multiplying half the slant side by the periphery of the base.

If the base be not circular, but any regular rectilinear figure, the solid is a regular **pyramid**, and the rule as to volume applies, substituting parallelopipedon for cylinder, *i.e.*

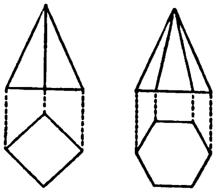


Fig. 20.

Volume = $\frac{1}{3}$ area of base × height.

Each side is a triangle and its area can be calculated separately, and hence the surface obtained.

EXAMPLES.

1. Find the weight of a boiler plate, 10 ft. 6 in. by 4 ft. 9 in., weighing 35 lbs. per sq. ft. *Ans.* 15 cwts. $65\frac{5}{8}$ lbs.

2. A triangular piece of plate, sides 3 ft. 6 in., 4 ft., and 4 ft. 6 in. respectively, weighs 1 cwt. Determine the weight per sq. ft. *Ans.* 17 lbs.

3. The cross-section of a canal is a trapezoid. Width at top 42 ft., at bottom 28 ft., depth 10 ft. Find its area. *Ans.* 350 sq. ft.

4. The sides of a field are AB 150, BC 42, CD 130, DE 72, and EA 50, all in yds. Also BE is $157\frac{1}{2}$, and BD 156 yds. There are no re-entrant angles. Find its area. *Ans.* 2.4 acres.

5. A boiler contains 530 tubes, $2\frac{3}{4}$ in. internal diameter. Find area for draught through them. *Ans.* 21.87 sq. ft.

6. Each tube in the preceding is 7 ft. long between tube plates, and 3 in. external diameter. Find the total heating surface. *Ans.* 2915 sq. ft.

7. The tube plates in (6) are $\frac{3}{4}$ in. thick, and each tube projects $\frac{1}{4}$ in. outside each plate. The tubes are of wrought-iron, weight of a cubic inch $\frac{5}{18}$ lb. Find the weight of one tube. *Ans.* 27 lbs.

8. A boiler is 10 ft. diameter, 12 ft. long; the water surface is at $\frac{3}{4}$ of the height. Find the volume of the steam space. *Ans.* 185 c. ft.

9. Find how many sq. yds. of non-conducting covering material will be required to cover the cylindrical portion, and one end, of the preceding boiler. *Ans.* 506.

10. The circular shell plates being 1 in. thick, and the end plates $\frac{11}{16}$; material mild steel, weighing the same as wrought-iron. Find the total weight of the shell, not taking laps or holes for furnaces into account. *Ans.* 8.65 tons.

11. Find the weight of a cast-iron hexagonal column, 12 ft. high, 8 in. outside across the corners, circular hole 5 in. diameter. Weight, $\frac{1}{4}$ lb. to 1 c. in. *Ans.* 788 lbs.

12. Construct a parabola on an 8-in. base, 3 in. high. 1st, symmetrical; 2d, with greatest height at 5 in. from one end.

13. Find the area of the preceding curves; by calculation, and also by Simpson's rule. *Ans.* 16 sq. in.

14. An elliptical door is 18 in. by 16 in.; material cast-iron; mean thickness $\frac{3}{4}$ in. Find its weight. *Ans.* $42\frac{3}{7}$ lbs.

15. A hollow steel propeller shaft is 10 ft. long over all; flange at each end 23 in. diameter, 4 in. thick; external diameter of plain part 14 in., diameter of hole 9 in. Find its weight. *Ans.* 3593 lbs.

16. The total depth of a cast-iron beam is 12 ins. The top flange is 6″ by $\frac{3}{4}$″, the bottom 9″ by 1″, and the web is $\frac{1}{2}$″ thick. Find its weight per foot length. *Ans.* $55\frac{7}{8}$ lbs.

PART I

THE PRINCIPLE OF WORK

CHAPTER I

MOTION—SLIDING, TURNING, AND SCREW PAIRS

WHEN two bodies shift their relative positions they are said to be in motion relatively to each other. Thus, a train shifts its position relatively to the earth; then the earth and the train are in relative motion.

The usual mode of expression in the preceding case would be to say that the train was in motion, but a little consideration shows that this is not a full statement. To a spectator on the earth, the train alters in position while the earth does not. Therefore he says the train moves. But to a traveller in the train it is the earth that moves while the train is still; so that, to be consistent, he should say the earth is moving. But actually he would still say that the train moved. If we examine more closely, we see that the reason of this clearly is that in each case the earth is tacitly treated as if it were a fixed body, which for all ordinary purposes it may be assumed to be. But now when we come to examine larger motions, as those of the heavenly bodies, we know that the earth is not treated at all as a fixed body, but is in motion relative to the sun and to all the other bodies. Questions relative to the solar system are sometimes treated as if the sun were fixed; but this again will not do when we have to consider the motions of the so-called fixed stars.

We see then that there is in nature no such a thing as an actual fixed body. And so we cannot speak of

the motion of one body only, but only of its motion relative to some other, the other body being either mentioned or else understood. The body most usually understood is the earth, and this explains why the spectator in a train speaks of himself as moving. But he would be perfectly correct if he said that the earth was moving past him. For the earth is moving past him just as really as he is moving along its surface.

The sort of relative motion which is possible between two bodies depends on the mode in which they are connected together. In some cases there is no restriction at all, *i.e.* when the bodies are entirely disconnected. They can then be moved relatively to each other in any conceivable manner. The bodies whose motions we shall consider are, however, generally so connected that only one kind of relative motion may be possible, either by being directly fitted to each other, or by being connected by intermediate pieces. The first kind of case being the simpler we take it in the present chapter, and we afterwards proceed to consider other cases in succeeding ones.

Sliding Motion.—The simplest kind of relative motion between two pieces is when one moves in a straight line relative to the other. Such a motion is called **Sliding**, and the pair of pieces is called a **Sliding Pair.**

The simplest example of this motion would be the

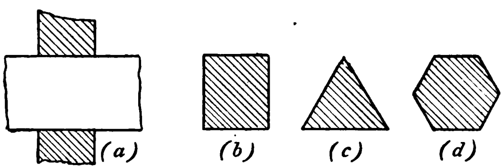


Fig. 21.

motion of a rod of any section, other than circular, in a hole which it exactly fits, as in Fig. 21, where (*a*) shows

a side view, and (*b*), (*c*), (*d*) are alternative sections. The bar in the cases shown is incapable of any other motion than sliding—which would not be the case were its section circular—and we have therefore a sliding pair.

A good practical example is found in the crosshead and guides of a direct acting engine. The crosshead may slide between two flat surfaces, as in Fig. 22, which shows a side view, the two guide surfaces being shaded; or the shoe may be so shaped, as in Fig. 23, as to fit into one guide only. The figure shows an end view. In both cases only the one motion—sliding—is possible.

Here we should notice that, although there appear in Fig. 22 to be two guides, this is only apparent. For if

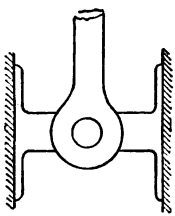


Fig. 22.

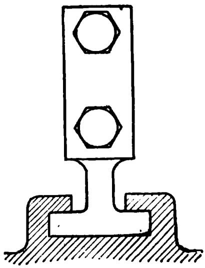


Fig. 23.

we examine more closely, we find that each of the guides is rigidly attached to the one frame of the engine, and so they are not two, but are only portions of one body, that body consisting of the whole of the framework and everything rigidly connected to it.

The second body in the relative motion is also more extended than appears at first sight. For it is not only the crosshead, but consists also of the piston rod and of the piston; the three forming one body, which slides relatively to the other body, consisting of the cylinder, guides, framework, etc., which are rigidly connected together.

So far the sliding motion has been the only possible relative motion, but bodies may move with a sliding motion even although other motions are not impossible. For example, a weight falling freely moves in a straight line—*i.e.* slides—relatively to the earth, although there is nothing to prevent sideways motion, because neither is there anything to cause it. More important still. A carriage or sledge of any description slides relatively to the earth, being kept to its proper direction by the pull of the traces or in a railway by the wheels. In all these cases there is perfect freedom of motion away from the earth if sufficient force be applied, but such force not being applied, the weight of the bodies acts as if a guide were fitted to the tops of the carriages, retaining them in contact with the ground.

It may be objected that, in the case of a carriage, the motion is not sliding but rolling, but this is not correct. The motion of the wheels is a rolling one, but these are not a rigid part of the carriage, but are attached to it simply to lessen the friction, and could, if the road were smooth, be dispensed with, as is the case in sledges. The motion of the carriage itself is unaffected by such fittings, being identical with that of a sleigh, which is evidently sliding.

Velocity.—We have in sliding a certain definite *direction* of moving, determined as we have seen; but we have also to consider in addition the *rate* at which the bodies move relatively to each other, or the **velocity** of one relative to the other—velocity being used to denote rate of moving. The term *Speed* is also used to denote the same thing.

In sliding, all parts of the moving body (assuming for convenience one body as fixed, and the observer, in all which follows, being supposed to stand on that fixed body) move in the same direction at the same speed, and the velocity of the body is measured by that of any point in it, *e.g.* in a carriage moving along a *straight*

road the shafts go exactly as far as the body, and one side as far as the other.

[This is only true on a straight road, which is the case of sliding; if the road be curved it no longer holds, as we shall see farther on.]

The velocity of a body depends on the space passed over, in a given time, by any point fixed in the body; and its measurement depends on whether the point passes over equal spaces in equal times—a case of **uniform velocity**—or passes over unequal spaces, when the velocity is said to be **accelerated**, if the successive spaces passed over increase; or **retarded**, if they diminish. Accelerated velocity often includes both cases, the acceleration being said in the latter to be negative.

To measure the velocity of a body moving uniformly, we note, at a given time, the position of a specified point in the body; and then, at the end of a convenient time, note again the position of the same point. Suppose this has been done, and the distance between the two positions measured and found to be s feet, the time taken being t seconds; then the distance moved in one second is s/t feet, or s/t feet per second is the velocity.

In the preceding paragraph it was not necessary to specify any particular instant at which to measure the velocity, but in dealing with non-uniform velocity we cannot speak simply of the velocity of the body, but only of its velocity at some particular instant. Such a velocity we cannot measure in the manner described, but we must in some way *estimate the distance the body would pass over in one second, if it moved uniformly during the second, at the rate at which it is moving at the particular instant considered;* if this distance be v feet, then we say the velocity *at the instant* is v feet per second.

The manner in which to make the estimate can only be determined with further knowledge than we possess

at present, but by anticipating somewhat, the method can be illustrated.

For example—a body in one second falls 16 feet, required its velocity at the end of the second.

In this case, the laws of Dynamics tell us that the velocity is a gradually increasing one, increasing at a uniform rate from zero to its final value, say v feet per second, *which we will for the future abbreviate thus,* v *f.s.*

Now in such a case the effect, on the whole, is the same as if the body had moved during the whole time, with just half the final velocity.

Therefore, the body if moving at $v/2$ f.s. for one second would cover 16 feet,

$$\therefore \frac{v}{2}=16, \quad v=32,$$

and the final velocity is 32 f.s.

Mean Velocity.—In the preceding example we see that—*If for the given time the body had moved uniformly with a certain velocity, it would have covered the same space which it actually did cover.* The certain velocity in the particular case above is $v/2$ f.s., but such a velocity can be found for all cases of motion, and is called the **Mean Velocity,** its definition being given in the italics.

Evidently, to determine the mean velocity during a given interval, we have only to measure the space traversed in feet, and dividing by the time in seconds, we get the mean velocity as above described in f.s.

Thus in the example given we simply say—Since 16 feet are traversed in one second the mean velocity is 16 f.s., caring nothing what may be the changes of velocity during the second.

We have now met for the first time with two conceptions of very great importance, viz.—varying quantities and their mean values. And we shall find that consider-

able aid is rendered to the correct comprehension of them by means of graphic representation.

Graphic Representation of Magnitudes.—When we have two magnitudes bearing certain relations to each other, we can represent these relations by means of a plane curve; values of one magnitude being marked off along an axis, and ordinates set up at the points so obtained, to represent the corresponding values of the other. A curve drawn through the tops of the ordinates exhibits to the eye the relationship between the magnitudes, and is often of great use in the solution of problems.

In the example previously considered the two magnitudes are Time and Velocity.

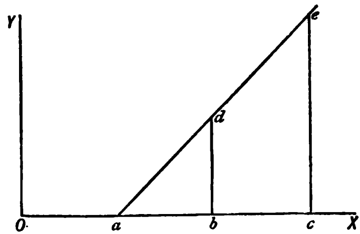


Fig. 24.

Take then two axes OX and OY.

Along OX we will set off time. Now it is quite immaterial at what point of OX we begin. Suppose then we let O, or time zero, represent $\frac{1}{2}$ sec. before the body was let go.

Mark off then O*a* to represent on some scale $\frac{1}{2}$ sec., O*a* is in the figure $\frac{1}{2}$ inch, the scale being 1 inch to 1 second.

At *a* the velocity is zero, so *a* is a point on the curve, the ordinate being 0.

Now set off $ac = 1$ second, *ac* will be 1 inch long.

Set up at *c*, *ce* to represent 32 f.s. It is in the figure

1 inch, so the velocity scale is 1 inch to 32 f.s. *e* is then a second point on the velocity curve.

We said that the velocity varies uniformly. This we express graphically by drawing the velocity curve as a straight line from *e* to *a*. For then the height of the curve from OX varies uniformly.

We can now by means of this curve determine the velocity at any instant during the second of motion, or we can determine the mean velocity.

To determine the mean velocity or the mean value of any magnitude, when we have in the manner shown represented it by a curve, we have only to determine the mean height of the curve. In the present case the mean height is plainly *bd*, where *b* is midway between *a* and *c*, and

$$\therefore \text{ Mean velocity} = bd,$$

and *bd* being $\frac{1}{2}$ inch, and the scale 32 f.s. to 1 inch,

$$\text{Mean velocity} = 16 \text{ f.s.}$$

We must be careful to notice that *b* represents mid-time of falling, not mid-height; so 16 f.s. is the velocity at the end of a half second.

Units of Velocity.—We have so far reckoned velocity in feet per second, because the foot and second are the most usual units of space and time. But any units whatever of space and of time may be used, depending on which are most convenient in any particular case.

It occurs rarely that a smaller unit than feet per second is necessary, but if so inches per second or feet per minute may be used. The latter of these is much used for reckoning piston speeds. In the motions of trains and ships it is customary to use larger units, and generally miles per hour is the unit selected.

For trains, and bodies on land generally, the mile is the ordinary mile of 5280 feet. Hence, if we wish to

interchange the units in expressing a velocity we use the following relation :—

$$
\begin{aligned}
1 \text{ mile per hour} &= 5280 \text{ ft. per hour,} \\
&= \tfrac{5280}{60} \text{ ft. per minute,} \\
&= 88 \text{ ft. per minute;}
\end{aligned}
$$

which is generally the other unit. If we require the velocity in f.s., then

$$1 \text{ mile per hour} = \tfrac{88}{60} = 1.46 \text{ f.s.}$$

In the case of ships there are two distinct miles used, viz.—

The ordinary mile as above, or
The nautical mile of 6080 ft.

Thus when the speed of a ship is given in miles per hour, it should be always stated which kind of mile is meant, since the difference is considerable. Working as before we have

$$1 \text{ nautical mile per hour} = \tfrac{6080}{60} = 101\tfrac{1}{3} \text{ ft. per min.;}$$

usually 101 is taken as sufficiently approximate.

The expression "nautical mile per hour" is never used in practice, but is abbreviated to **knot**. So that in the preceding we should write

$$1 \text{ knot} = \tfrac{6080}{60} = \text{etc.}$$

It is common to find "knot" used as if it represented "nautical mile," so we have speeds given as 14, 15, etc. "knots per hour." This is erroneous, and should be guarded against, not that in the present case any grave error is caused, but because the student, by never using a term in any other than its strict meaning, will save himself from falling into numerous difficulties.

One effect of the difference in length of the nautical and land mile is to create a false impression regarding speeds of ships. Thus if a ship have a speed of 20 knots, then

$$
\begin{aligned}
20 \text{ knots} &= \frac{20 \times 6080}{5280} \text{ miles per hour,} \\
&= 23 \text{ miles per hour,}
\end{aligned}
$$

which *appears* a much higher speed.

Turning.—We consider next another simple kind of relative motion, viz. That in which two bodies are so connected that one can only move by turning round a centre fixed in the other and *vice versa.* The motion we call **Turning,** and the two bodies form a **Turning Pair.**

The simplest example of a turning pair is a round rod fitting in a hole (Fig. 25). The rod has collars

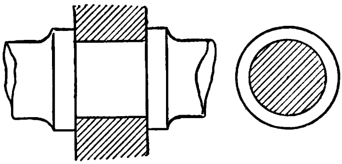


Fig. 25.

which prevent endwise motion, and thus turning is the only relative motion possible. As practical examples we may take a propeller or crank shaft (Fig. 26) in its

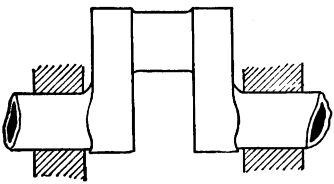


Fig. 26.

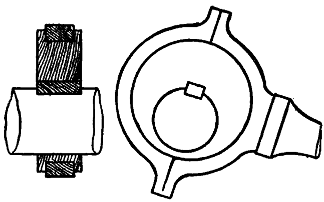


Fig. 27.

bearing, an eccentric and strap (Fig. 27), or a wheel on its axle.

It appears in the case of a propeller shaft endwise motion is possible. But this is prevented by the thrust bearing at another part of the shaft. In crank shafts the journals, or parts of the shaft in the bearings, are sometimes turned smaller than the rest of the shaft, as Fig. 28, but it is best in marine engines to leave this also to the thrust block, such journals being nearly

always a source of trouble. In turning, as in sliding, we often meet cases in which the motion is not strictly defined by the connection of the pieces. For example, a heavy shaft with the caps off the bearings would still revolve in them, being kept in place by its weight, and they form a turning pair (compare page 20). The only conditions the bodies must satisfy are that a circular projection on one must fit a hole in the other; and motion other than turning must in some, it matters not what, way be prevented.

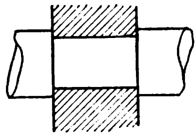


Fig. 28.

In estimating the motion of a sliding piece we were at liberty to select any point on the piece, the motion of all points being identical. But this is no longer the case in a turning pair. For example, let C be a point in the moving body, the paper representing the fixed one; O is the centre of motion, *i.e.* the centre of the pin on the moving body and the hole in the fixed one, or *vice versa*.

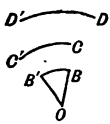


Fig. 29.

Then during the motion C moves say to C′ in the circular arc CC′. Now the motion of C is not sliding, since CC′ is not a straight line. But, taking for simplicity the case of uniform motion, we can take the curve as made up of a large number of small straight pieces, along each of which C slides in turn at a constant velocity, continually changing the direction of the velocity but not its magnitude. Such motion, although not sliding, can yet be measured in the same units as sliding; and we say the *Linear Velocity* of C is given by dividing the length of the arc CC′ by the time occupied in describing it.

C then moves at say v f.s., the instantaneous direction of its motion at any point of the arc being along the small piece of arc at that point, *i.e.* along the tangent.

But now this velocity of the point C does not give us the velocity of the body. Because, for example, the

point O does not move at all; then a point B, nearer to O than C is, moves slower than C; while one D outside moves faster.

We require then some different method, and we proceed thus :—

Instead of considering the motion of the point C, let us draw the line OC of indefinite length and consider how that moves. The line starts at OC, and as C moves to C′ it turns to OC′, turning through the angle COC′.

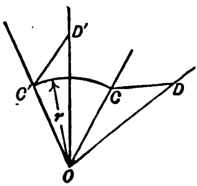


Fig. 30.

Now it does not matter what line in the body we take, we shall find that they have all turned through the same angle.

Take first another line OD through the centre, then OD turns to OD′, while OC turns to OC′. Thus C′OD′ is only COD in a new position,

$$\therefore <C'OD' = <COD,$$

add to each $<COD'$, and we have

$$<C'OC = <D'OD.$$

Next, take any line whatever, CD represents such a line —C and D being any points in OC, OD.

Then the triangle C′OD′ is COD in a new position; and evidently during the motion each side must turn through the same angle.

We see now then that we can completely define the motion of a turning piece by giving the angle turned through by any line in it.

Angular Velocity.—When a point moves it traces out a line, and its velocity is measured by the length of line traced out in a unit of time.

When a line swings round a point, as OC (Fig. 30), it traces out an angle, and so we define its velocity by the angle which it traces out in a unit of time. The

angle traced out per unit time gives us then the **angular velocity** of the line, and therefore also of the body on which it is drawn.

The relative velocity of a turning pair is then an angular velocity, and is, if uniform, measured by the angle turned through per second. The measurement of a non-uniform velocity has been fully explained in the case of linear velocity, and need not be repeated (pages 21 and 22).

Units of Angular Velocity.—Angle is measured either in English measurement by Degrees, of which 90 form a right angle; in French measurement by Grades, the right angle being divided into 100 parts; in practical work by revolutions or whole turns.

[In this case the student must remember that the angle does not simply refer to the space between the old and new positions of the swinging line, but to the whole space which has been swept out by the swinging line since it commenced swinging. Thus in Fig. 30 OC might go on revolving for a number of turns and finally arrive at OC′, the angle traced out would then be C′OC plus the whole of the turns.]

Or, which is best of all for all purposes, by circular measure, which we will now explain. Referring to Fig. 30, let OC be the swinging line; take C, any point in it, then C moves in a circle as OC swings; let OC swing to OC′, tracing out the angle C′OC. Then we measure C′OC by the ratio which the arc CC′ bears to the radius OC,

$$\therefore \text{ Circular measure of } C'OC = \frac{CC'(\text{arc})}{OC}.$$

This measure is independent of the position of C. For evidently

$$\frac{\text{arc } DD'}{2\pi \times OD} = \frac{\text{arc } CC'}{2\pi \times OC}.$$

Each denominator representing the whole circumference of its own circle;

$$\therefore \frac{\text{arc } DD'}{OD} = \frac{\text{arc } CC'}{OC} = \frac{\text{arc}}{\text{radius}} \text{ generally,}$$

and whether we take D or C, or any other point, we get the same numerical value of the circular measure.

It must be clearly understood, however, that in this method of measuring angles we only differ from the other methods in the size of the unit angle. Angles can only be measured in terms of angles, but the unit chosen in this case has certain advantages in simplifying formulæ which makes it superior to the others. Its one disadvantage, if it be one, is that it is large, so that fractions have to be used.

We must now see what this unit angle is.

Unit of Circular Measure.—Being the unit, its value is 1,

$$\therefore\ 1 = \frac{\text{arc}}{\text{radius}},$$

so that the unit subtends at the circumference an arc equal in length to the radius.

This properly defines it, and enables us to compare it with our other units. For example—1° or 1 degree subtends an arc, whose length is $\frac{\pi \times \text{radius}}{180}$, there being 180 degrees in the half circumforence,

$$\therefore\ \frac{\text{Unit of degree}}{\text{measure}} : \frac{\text{unit of circular}}{\text{measure}} = \frac{\pi \times \text{rad.}}{180} : \text{rad.},$$

whence

$$\text{Unit of circular measure} = \frac{180}{\pi}\ \text{degrees},$$
$$= 57 \cdot 3^\circ.$$

The name *Radian* has been given to this unit.

Relation between Angular and Linear Velocity.—By expressing angular velocity in circular measure we can obtain a simple relation between the angular velocity of a turning piece and the linear velocity of any point in it.

For, referring to Fig. 30, let C be the point at a radius $OC = r$ say; let $V =$ linear velocity of C; A

= angular velocity of body in circular units. If now t be the number of units of time taken in moving through COC′,

$$CC' = Vt,$$

and

$$\angle COC' = At.$$

But

$$\angle COC' = \frac{CC'}{r} \text{ circular units,}$$

$$= \frac{Vt}{r},$$

$$\therefore At = \frac{Vt}{r}, \quad \therefore A = \frac{V}{r},$$

or

$$V = Ar.$$

Thus we have a simple relation between the two velocities.

One important use of the preceding is to determine the rubbing velocity of a shaft in its bearing. r is then the radius of the bearing, and if r be in feet and A in radians per second, V gives in feet per second the velocity with which the metal of the shaft rubs over that of the bearing.

Radius of Reference.—The above also shows us that although the linear velocity of any point C is not sufficient to determine the turning velocity, yet when combined with a statement of the radius at which C is, it is sufficient. The velocities of turning pairs are often stated in this way by giving the linear velocities of points at a certain radius, the radius selected being called the **Radius of Reference.** Evidently we can in this way compare the velocities of turning pairs, or even of a turning and a sliding pair.

Degrees or grades are never practically used to measure angular velocity, but revolutions per minute is a common unit of measurement, so we must compare this method with the circular unit measurement.

We have then—

$$1 \text{ revolution} = \frac{\text{circumference}}{\text{radius}},$$

$$= 2\pi \text{ units of circular measure};$$

$$\therefore\ 1 \text{ revolution per minute} = 2\pi \text{ circular units per minute},$$

$$= \frac{2\pi}{60} \text{ circular units per second};$$

$$\therefore\ n \text{ revolutions per minute} = \frac{2\pi n}{60} \text{ circular units per second},$$

or, if A be the circular measure of the same velocity reckoned per second,

$$A = \frac{2\pi n}{60}.$$

Revolutions per second is not a measure of common occurrence.

Screw Motion.—The two kinds of motion we have just investigated can be represented on a plane; because, although the bodies dealt with have been solid, yet parallel plane sections of them each moved in its own plane, and any one plane section could be taken to fully represent the motion of the whole solid.

The third simple case of motion, viz. Screw Motion, which we are about to consider, consists of motions in perpendicular planes, for while a section perpendicular to the screw axis revolves in its own plane, it also advances along the axis.

The simplest case is that of a common bolt and nut shown in Fig. 31. When we turn the bolt head in the direction of the arrow (*b*), there ensues, beside the turning motion, a forward motion, *i.e.* motion to the right, in (*a*).

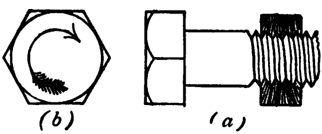


Fig. 31.

We may state here that we shall find it convenient to use terms which define the direction of a turning

velocity, by comparing it with the motion of the hands of a clock. The arrow in Fig. 31 shows *clockwise* turning, or right-handed turning, and turning as here shown  is called anti-clockwise or left-handed.

Screw motion then is compounded of turning and sliding. We know how to measure each of those separately, and we want now to consider what relation holds between the two. For this purpose we must consider how a screw thread is formed.

To cut a screw, a plain cylinder is put on the lathe, as in Fig. 32, which shows a plan and section, C being

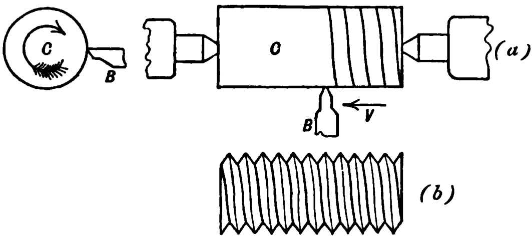


Fig. 32.

the cylinder. This cylinder turns uniformly. A tool B is meanwhile pressed against it, and the saddle moved along the bed of the lathe, carrying with it the tool, also at a uniform speed.

The effect is that the tool cuts a spiral groove as shown in (*a*).

The operation is continued over and over again, a deeper cut being taken each time, till we obtain the shape shown in the lower figure (*b*), a final smoothing cut being given by a chaser. Let now

A = Angular velocity of C,
V = Velocity of tool,
p = Pitch of thread, *i.e.* the distance between consecutive threads.

Then while the cylinder turns once, the tool moves p,

$$\therefore p = \mathrm{V} \times \text{time of one revolution},$$

$$= \mathrm{V} \times \frac{2\pi}{\mathrm{A}},$$

or

$$\frac{\mathrm{V}}{\mathrm{A}} = \frac{p}{2\pi}.$$

In cutting the thread we forcibly move the tool at the speed V. But if now, having cut the thread, we hold the point of the tool in the groove, leaving it free to move along the lathe bed, and then rotate the cylinder at the speed A, it follows that the tool will be moved along at the speed V. And evidently this will still hold, if instead of the solitary point of the tool, we insert in several threads of the screw the corresponding threads of a nut. Each piece of the thread moves as the tool point would, and hence the whole nut will move at the speed V given above.

We have in the preceding assumed the cylinder rotated, and the nut to move along its axis. Evidently if the nut were held and the cylinder rotated it would move at the speed V through the nut, V being simply the relative velocity and therefore the same whichever is fixed. V is here called the Speed of Advance. Also the cylinder may be left free to rotate, and then if the nut be moved at the speed V, not being allowed to rotate, the cylinder must rotate at the speed A.

[The last motion is not always possible, on account of friction. This, however, we need not consider at present.]

Double, etc. Threads.—In Fig. 32 the pitch p is comparatively small, so that the edges of the grooves nearly meet when the one spiral is cut, without cutting to any great depth. This screw has then a single thread or spiral.

If, however, we require V to be large compared to A, or if the screw be one rotated by the endwise motion of

the nut, then p will be large, and since we cannot cut the groove very deep owing to the weakening effect on the screw, the cylinder when one spiral or thread had been cut would appear as in the upper figure of Fig. 33.

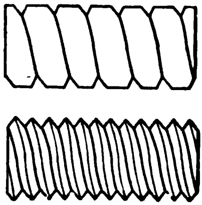


Fig. 33.

The screw would not, however, be used in this state, but the tool would be shifted to midway between the grooves, and a second groove cut, thus obtaining the shape shown in the lower figure. The screw appears at first sight like Fig. 32, so far as the edges are concerned, but on looking at the angle of the threads the difference is plain.

We have now practically two screws, each of pitch $2p$, and two threads must be cut in the nut. The nut moves as it would do on each alone, but there is twice as much bearing surface and hence less wear.

The screw thus cut is said to have a double thread, and we may extend the process to three or more separate threads.

The velocity ratio is, as before,

$$\frac{V}{A}=\frac{\text{pitch}}{2\pi},$$

the pitch being that of either spiral, *i.e.* in Fig. 33, $2p$ not p.

An important example is that of a screw propeller.

Screw Propeller.—This consists of a very short piece of a screw, working, not in a solid nut, but in the water, which acts similarly to a nut, though with some important differences not needful here to consider.

If the propeller worked in an actual solid nut at an angular velocity A, then it, and consequently the ship to which it is attached, would advance at a speed V given by

$$V=A.\frac{p}{2\pi}.$$

But the water being yielding, it is as if the nut slipped back, and the actual speed is less than V by a certain amount, called the slip. This effect belongs to Hydraulics, so now we only mention it, but shall not take it into account.

If now the engines be revolving at N revolutions per minute, and p be the pitch, in feet, of the propeller, we have, neglecting slip,

$$V = A\frac{p}{2\pi} = \frac{2\pi N}{60}\cdot\frac{p}{2\pi},$$
$$= \frac{Np}{60}\text{ f.s.},$$

a result which we can also obtain directly, since at each revolution the ship advances p feet.

The pitch being coarse, the propeller will, when the ship is under sail and the engines not working, be revolved by its passage through the water if it be not held fast.

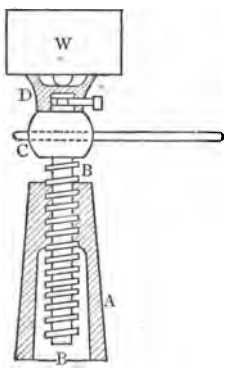


Fig. 34.

Another practical application of the screw is the Lifting Jack.

This is simply a screw, the nut A of which rests on the ground, and the screw B can be revolved by a handle passing through the head C, which is solid with B. When the handle is turned, B screws out and lifts a weight W placed on the top of the piece D, which moves longitudinally with B but does not turn with it, this being effected by cutting a circular groove in the end of B which fits in D, and screwing the end of a small set screw into the groove. The pitch being small compared to the circumference of the circle described by the end of the handle, the longitudinal

motion is slow, which, as we shall see hereafter, implies that the screw can exert a heavy thrust.

As an example of the reversed action, we may take the common Archimedean Drill.

A screw of very long pitch is cut on the spindle of the drill, and the nut being moved alternately backward

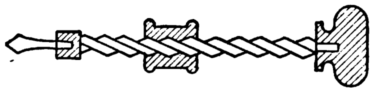


Fig. 35.

and forward, the drill rotates alternately in opposite directions.

EXAMPLES.

1. A train is running at 20 miles per hour. Find its velocity in f.s. *Ans.* $29\frac{1}{3}$.

2. If in (1) the train be stopped in 10 seconds, its velocity being decreased uniformly by the brake, at how many yards from the stopping point were the brakes applied? *Ans.* $48\frac{8}{9}$.

3. The stroke of an engine is 2 ft. 3 ins., and it runs at 130 revolutions per minute. Find the mean speed of piston, and the angular velocity of the crank shaft (supposed uniform).
Ans. 585 ft. per min.; 13.6 per sec.

4. The piston speed of an engine is 850 feet per minute. Supposing the velocity curve (p. 24) during the time of one stroke be a semicircle, find the maximum speed of piston; and also the speed when $\frac{2}{3}$ of the time of a stroke from the commencement have elapsed. *Ans.* 1082 and 1067 ft. per minute.

5. The stroke of an engine is 4 ft., revolutions 96 per minute, diameter of crank shaft 14 ins. Determine the speed of rubbing of the main bearings, and compare it with the mean value of the rubbing velocity of the crosshead guide.
Ans. 5.87 f.s.; ratio, 11 : 24.

6. Cast-iron should be cut at from 12 to 16 feet per minute. The travel of a planing machine is 3 ft., and it makes the return stroke at twice the speed of the cutting one. How many strokes per minute should it make when planing cast-iron?
Ans. $2\frac{2}{3}$ to $3\frac{5}{9}$.

7. Brass should be cut at 25, and wrought-iron at 22 feet per minute. Find the revolutions a lathe should turn at: 1st, when turning a brass plug 2 inches diameter; 2d, when turning a $\frac{1}{2}$-in. wrought-iron pin.

Ans. 48 per min. nearly; 168 per min. nearly.

8. A thread is being cut on a 1-inch brass screw. Find the proper angular velocity of the work, and also the velocity at which the tool should travel to cut the thread. If the saddle be moved by a screw of $\frac{1}{2}$-inch pitch, how many revolutions should it revolve at?

Ans. 10; .2 ins. per sec. very nearly; $23\frac{19}{22}$ per minute.

9. The pitch of screw in a screw jack is $\frac{7}{16}$ ins., and it is turned by a handle 19 ins. long. Compare the speed of the end of the handle to that of lifting. *Ans.* 273 : 1.

10. A ship moves at 17 knots. Find her speed in f.s. and in miles per hour. *Ans.* 28.7; 19 very nearly.

11. If in (10) the propeller pitch be 16 ft. Find how many revolutions the engines run at, neglecting slip. *Ans.* 107.3.

12. In (11) the thrust rings on the shaft are 2 ins. wide, $14\frac{3}{4}$ ins. external diameter. Find the maximum and mean rubbing velocities over the surface. *Ans.* 6.91 and 6.44 f.s.

CHAPTER II

EFFORTS AND RESISTANCES—FRICTION

THE relative motion of a pair is, in nearly all cases, resisted by some force, which we hence call the **Resistance.** And in order to produce the motion a force must be applied, which we call the **Effort.**

For example, consider the sliding pair consisting of the piston and cylinder of a steam engine.

Then the relative motion is resisted, the resistance being supplied by the connecting-rod end which bears against the end of the piston rod. The relative motion then will not take place until a sufficient effort has been applied to the piston by the steam.

We therefore now inquire into the sources from whence we derive our Efforts and Resistances in nature, and also into the way in which we are going to measure these magnitudes.

A source of effort or energy must be capable of exerting a force and of following it up. Of such sources the principal is—

Elasticity of Fluids.—By fluids we must not be understood as meaning liquids, the term fluid including liquids, gases, and vapours. It is of the latter two we speak, and they may be spoken of as elastic fluids, in distinction to liquids.

The elasticity of a fluid is the name by which we denote the power it possesses of exerting pressure on the sides of the vessel in which it is contained. If this

pressure be greater than that on the outside, then the sides, if elastic, will expand; or, more usually, the fluid is contained inside a vessel, as a cylinder, in which fits a piston, free to move, as Fig. 36.

If now the elasticity of the gas be greater than the outside pressure, the difference of the two supplies an effort, which will move the piston out against a resistance, the effort exerting energy, and work being done on or against the resistance.

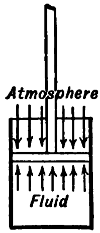


Fig. 36.

If we have simply a cylinder containing a definite quantity of fluid, then this process will come to an end by gradual decrease of the elasticity till it becomes only just sufficient to balance the outside pressure. But if now we apply heat to the fluid, then its elasticity will be kept up, and it will continue to exert an effort and drive the piston forward to an extent limited only by the length of the cylinder and the supply of heat. Here then we see such elasticity is one of our chief sources of effort, being the means by which the energy called Heat is utilised. For a study of the actual processes the student must consult treatises on heat engines.

The elasticity of solid bodies can also furnish us with efforts. For example, a spring wound up furnishes the effort which drives a watch.

Gravitation is our other great source of effort. Thus water falling on high ground gravitates downward, and may be used to give an effort; either by its weight being collected in an elevated reservoir, and allowed to descend on to the buckets of a water wheel; or by its motion in a stream driving the vanes of a wheel dipping in it; or where a great fall is available it may drive a turbine.

The original source of the effort is, in these cases, traceable to the heat of the sun, which originally raised the water in the form of vapour, and this is why water

is nearly always the medium whence gravitation efforts are derived. If the effort be derived from the action on a solid body we must first have lifted that body up, *e.g.* the weights of a clock or a pile driver. A similar remark also applies in the case of the elasticity effort when derived from a solid body, *e.g.* the watch spring must first have been wound up.

Again, the sun heat produces air currents, which exert efforts on the sails of windmills.

Lastly, we have the muscular efforts of living beings.

In all cases the effect is—We have a force exerted driving a piece of some kind before it, producing relative motion between that piece and some other which forms a pair with it. The motion may be sliding, as in the piston and cylinder, or turning, as in the water wheel and its bearings.

Next, what are the chief resistances we meet with?

The first answer to this is that the sources of effort are also sources of resistance, for taking them in order—

Elasticity of a fluid or of a solid body furnishes the resistance when the work to be done is the alteration of volume of the fluid, or of shape of the solid body; *e.g.* the compression of air in a cylinder, the elasticity of the air resisting the sliding of the piston in the cylinder; or the elasticity of a safety-valve spring resisting the upward movement of the valve relative to its seating.

Gravitation again is perhaps the chief source of resistance, since in nearly all work the lifting of weights forms a large part, and in the raising of a weight gravitation directly resists the motion. It also is a great indirect source of resistance by causing friction.

The forces then whose sources we have considered may be equally well efforts causing, or tending to cause, relative motion; or resistances, tending to stop it. And hence they are classed as **Reversible Resistances**.

One kind of resistance, viz. that due to inertia, we

mention for completeness, but for the present leave out of account.

But there is now an important class of resistance which differs essentially from the preceding, viz. **Friction or Frictional Resistances.**

Let us consider the motion of a sliding pair, consisting of a heavy body A on a horizontal table C, the body sliding in a guiding groove.

First, let a spring B bear against A and against a stop on the table.

If now we move A towards the stop the elasticity of B supplies the resistance to the relative motion of A and C. But when we have thus compressed B its elasticity can supply an effort causing relative motion of A and C. Elasticity then is reversible, and can supply either the effort causing or the resistance resisting the relative motion of the pair of bodies on each of which it acts. But now let there be no spring B, but suppose, which we have neglected in the preceding, that the table is rough. Then we know that if we move A towards the stop, the friction between A and C will supply a resistance to the motion. But if now, having moved A, we let it go, the friction will never move it back. And, moreover, if we now try to move A back the friction will offer just as much resistance to the return motion as it did to the first.

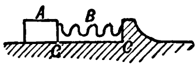


Fig. 37.

The friction then between two pieces of a sliding pair never tends to produce relative motion of the pair, but always to prevent it. The same holds true for all kinds of pairs, and hence we term friction an **Irreversible Resistance.**

Measurement of Force.—We have next to consider how to measure our efforts and resistances, *i.e.* forces, generally.

Force, like all other magnitudes, can only be meas-

ured in terms of a magnitude of the same kind, so that among our known forces we must pick out one in terms of which to measure all the rest.

The force which we select for this purpose is one due to gravitation, and the system of measurement is accordingly called the Gravitation System—the unit force being called the Gravitation Unit.

Gravitation Unit.—This unit force is the force exerted by gravity *on*, or the weight *of*, a certain lump of platinum kept in the Exchequer Office in London, and defined as one pound.

Our unit force then is the weight of one pound, and we measure forces in terms of this unit, or in ordinary language, in pounds. Thus, for example, if we hang up to a peg a piece ten times as heavy, we should say the pull on the peg is 10 pounds.

Now, there is a theoretical drawback to the use of the gravitation unit, which is, that the force exerted by gravity on the lump of metal is not a constant one at different parts of the earth's surface, being at the poles $\frac{177}{176}$ times as much as at the equator. We thus have a variable unit, and we should, for definiteness, insert the particular position on the earth's surface at which the force is to be measured. If this be done we have a quite definite unit.

For scientific purposes this may be done, but usually for such purposes another unit is used, depending on the known laws governing matter and motion. For the purposes of the engineer, however, it is quite unnecessary to consider such refinements, since that absolute accuracy which must in scientific matters be attained, is not only not necessary, but cannot possibly be obtained in results which depend for their accuracy on that of the instruments by which they are obtained.

Suppose, for instance, a column 4 inches in diameter to be sufficient to support certain material at the equator, what difference should there be at the pole ?

The area should be increased in the ratio $\frac{177}{176}$, or the diameter in the square root of this ratio,

$$\therefore \text{ Correct diameter at equator} = 4 \times \sqrt{\tfrac{177}{176}},$$
$$= 4.012.$$

Now it is not at all unlikely that such a column though intended to be 4 inches in diameter, would be quite 4.012 inches, this being quite within ordinary limits of accuracy, and, even if not, there would have been an ample margin of strength allowed in the original design to admit of such a small increase of load without prejudicial effect.

For absolute accuracy of expression also we should not speak of a force of 10 or 20 pounds, but of 10 or 20 pounds' weight, since the unit is not the lump of platinum but its weight. No harm, however, will ensue, and time is saved by the abbreviation, so long as the student clearly comprehends that the unit is the force or weight, and not the lump of metal.

We will now see then how to measure the various forces in order.

Elasticity of a Fluid is measured by allowing it to push out a piston of known area, compressing a spring before it.

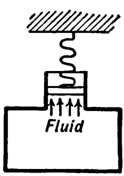


Fig. 38.

We then measure the compression of the spring, and knowing by experiment the weight which will compress it to the same extent, we know the amount of the effort or pressure on the piston.

For example—let the piston be 2 sq. ins. in area, and let the spring be found compressed to the same extent as would be done by a force of 60 lbs. Then

$$60 \text{ lbs.} = \text{total force on piston.}$$

The force on any other part of the surface is proportional to the area, and we most easily find it by first finding the pressure on each square inch. *The elasticity*

is then measured by the pounds pressure on a square inch.

In the present example

$$\text{Pressure on 1 sq. in.} = \frac{60}{2} = 30 \text{ lbs.}$$

and we say the fluid is at 30 lbs. per sq. in. pressure.

Then the pressure on any number, say n sq. in. of the surface, is 30 n lbs.

EXAMPLE.—Steam is admitted to a cylinder 20 ins. diameter at a pressure of 30 lbs. by gauge; what is the effort on the piston? The pressure is "30 lbs. by gauge"; here "per sq. in." is omitted, which is common in actual practice; also we must ask—Does "30 lbs. by gauge" mean that the actual pressure of the steam on a surface in contact with it is 30 lbs. on the sq. in.? The answer is that it does not. For a boiler pressure gauge is so marked as to show, not the pressure, but the difference between the steam pressure inside and the air pressure outside. This is common to most pressure gauges. For another case we have so-called vacuum gauges, which are attached to spaces in which the pressure is less than that of the atmosphere; these show the amount by which the inside pressure falls short of the outside.

The actual pressure on the inside surface is called the Absolute Pressure. We have then

$$\text{Absolute pressure per sq. in.} = 30 + \text{atmospheric pressure.}$$

The latter varies, and must for any given case be measured by the barometer at the particular time considered. It does not, however, vary much from 14.7 lbs. per sq. in., which value can generally be taken as quite accurate enough,

$$\therefore \text{Absolute pressure} = 44.7 \text{ lbs. per sq. in.}$$

and

$$\begin{aligned} \text{Effort} &= \text{area in sq. in.} \times 44.7, \\ &= \frac{\pi}{4} \times 400 \times 44.7, \\ &= 14{,}043 \text{ lbs.} \end{aligned}$$

When we have large forces as here to deal with, we often use a larger unit, viz. the **Ton** of 2240 lbs.

Thus in above

$$\text{Effort} = \frac{14043}{2240} = 6.027 \text{ tons.}$$

Gravitation Efforts, etc.—These are, of course, the easiest of measurement, although we took Elasticity of Fluids first, as we had before given it the first place.

To determine the resistance gravity offers to the lifting of a body, we can either actually weigh it in a weighing machine, or if this be not convenient or possible, *e.g.* when we have to estimate the weight of a body from the drawing of it before it is made, we calculate its volume; and then knowing the weight of a known volume of the material of which it is composed, we can easily deduce its weight. Thus the resistance offered to the lifting of a boiler plate 12 ft. by 6 ft. by 1 inch is found thus—

$$\begin{aligned} \text{Volume} &= 12 \times 6 \times \tfrac{1}{12} \text{ c. ft.,} \\ &= 144 \times 72 \times 1 \text{ c. in.} \end{aligned}$$

Now 1 c. in. weighs $\frac{5}{18}$ lbs.,

$$\begin{aligned} \therefore \text{ Weight} &= \overset{8}{\cancel{144}} \times 72 \times \frac{\overset{40}{\cancel{5}}}{\cancel{18}}, \\ &= 2880 \text{ lbs.} \end{aligned}$$

[The work is facilitated by leaving all the arithmetic to the last, unless the intermediate results be required. For example, the volume above is 10368 c. ins., but since we do not require it we shorten the work by not calculating it out.]

Laws of Resistance—Spiral Spring.—The resistance offered by a spiral spring to extension or compression can, for any given spring, be determined for any given alteration of length by determining, by actual experiment, the weight which will cause the particular alteration. Similarly during the compression of a given volume of fluid behind a piston, the resistance at any instant can be determined by a pressure gauge. It is found, however, that the resistances so found are, in the first example, connected by a certain law with the alterations in length of the spring, and in the second connected by a similar law with the change of volume.

Thus knowing one value of the resistance we can determine any others we require. Take first the spiral spring.

The figure shows a spring in three positions. In the centre it is in its natural state, length l, while in the other two it is represented as compressed and extended respectively through a distance x.

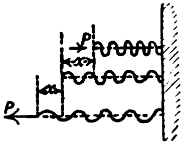


Fig. 39.

The law then is—

If x be small compared to l, then the force P required, either to compress or extend the spring, varies directly as x. We may say

$$P = P_0 \frac{x}{l},$$

where P_0 is the force which would double the length or compress it to zero, *if the law held good*, but this it does not do when x becomes large.

We can conveniently represent the law graphically thus—

Take OA $= l$ and produce it.

[The line is broken since we want to use a fairly large scale.]

Take points 1, 2, 1′, 2′, etc., and at each point set up an ordinate representing, on a selected scale, the force required to compress or extend the spring to the said point.

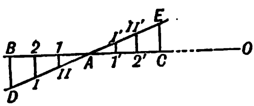


Fig. 40.

We thus get a curve DAE through the tops of the ordinates, and the law says that so long as we do not go too far from A the curve is a straight line. What its shape is farther away we do not discuss. To prove that this agrees with the law, we have by similar triangles

$$\frac{BD}{AB} = \frac{2I}{2A} = \ldots = \frac{I'1'}{A1'} = \ldots = \frac{CE}{AC},$$

i.e. the force varies as the extension or compression in each case.

Elastic Fluid.—The magnitude of the effort exerted,

or the resistance offered by a certain mass of elastic fluid during expansion or compression, varies according to the circumstances, and the law can only be determined by aid of the principles of Thermodynamics, and then only for certain simple cases. There is, however, one law which by reason of its simplicity is most used, and which applies very nearly to many actual cases.

The law just mentioned is known as Boyle's Law, and is as follows: The pressure per sq. in. exerted by a given quantity of a fluid on the sides of the containing vessel varies inversely as the volume of the vessel.

By *quantity* we mean weight, not volume, because any quantity of an elastic fluid, however small, will, if allowed, expand, and fill any volume however large.

Thus if we have in a cylinder fluid at 90 lbs. pressure, then by moving the piston till the volume is halved, the pressure will be doubled and *vice versa.*

This law also is well suited for graphic representation.

The original pressure and volume being given, we proceed as follows:—

Choose a scale for volumes, *i.e.* 1 inch to represent say n cubic feet; and a scale for pressures, *i.e.* 1 inch to say m lbs. per square inch.

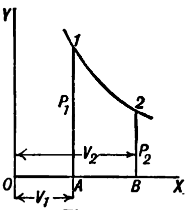


Fig. 41.

Set off OA on the volume scale to represent V_1, the original volume in cubic feet, then $OA = V_1/n$ ins. And also A1 to represent P_1, the original pressure, then $A1 = P_1/m$ ins.

Let now the piston move until the volume is V_2 cubic feet, and the pressure P_2 lbs. per square inch. Set off

$$OB = V_2, \quad B2 = P_2,$$

and so on for a number of other volumes.

We shall thus obtain a number of points similar to

1 and 2, and we then draw a curve through these points as shown.

The curve so drawn represents the relation between the pressure and volume of the given quantity of fluid.

Now the law is

$$P_1V_1 = P_2V_2 = \text{any } P \times \text{the corresponding } V,$$

i.e.

$$A1 \times OA = B2 \times OB = \text{any ordinate} \times \text{the corresponding abscissa.}$$

Looking back to the Preliminary Chapter, we see that this is a known curve, viz. the Hyperbola.

Hence, then, we need only know one point, say 1, on the curve, and we can then construct it by the method given (page 8), and thus determine graphically the pressure at any given point of the expansion or compression.

The law then gives us, at any point, the pressure per square inch, and we calculate the effort as already explained.

In an actual steam engine cylinder, we cannot find the law of effort, yet we can by means of an instrument called an Indicator make the pressure register itself, and actually draw a curve of the nature of the one we have just been considering. We shall return to this important question in the next chapter.

Friction.—We will now see in what way to calculate the values of resistances due to friction.

We shall treat for the present only the sliding pair, leaving the turning pair until we have inquired into the peculiar character of the efforts and resistances in that kind of pair. Also, we leave the case in which one of the bodies is a fluid—*e.g.* a ship sliding relative to the water—to the section on Hydraulics.

Whenever we move one element of a pair relative to the other, the surfaces being pressed together, a certain resistance is offered to the motion; which resistance we know varies with the state of the surfaces, and also with

the force with which they are pressed together. This resistance is called Friction.

The foregoing facts are matters of common knowledge. For example, let us take a sledge loaded with a certain load; then if the road be fairly smooth, the sledge can be drawn along; and the greater the load the greater the force required to draw it. If the road, however, exceed a certain amount of roughness the sledge would not move at all; while, on the other hand, if we smooth the road, less and less effort is required, till, when we come to a smooth surface such as ice, the effort becomes very small indeed.

Now we do not in this country use sledges, but the example is taken because the surfaces rub on each other, and the sliding friction is evident. The friction in the case of a wheeled carriage is of a more complicated nature if we examine it thoroughly; but just as the motion is, on the whole, sliding; so, on the whole, will the friction follow much the same law as sliding friction.

The surfaces we shall principally have to deal with are not like that of a road, but are of metal, and made as smooth as circumstances will admit of.

What we wish to know now then is, What is the law, if any, which connects together the resistance offered and the pressure between the surfaces? And how does it vary for different surfaces?

Experiment only can furnish the answer required, and accordingly experiment has been used with the following results:—

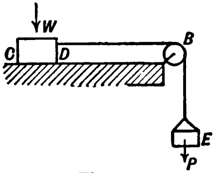


Fig. 42.

Let CD represent a small sledge, which can be loaded as required, this sledge slides on a path on a horizontal bed. Attached to CD is a cord passing over a pulley at B, and having on its end a box E, which can be loaded. Suppose now the sledge be

loaded with weights which, together with its own weight, come to W lbs. Then *to keep the sledge moving uniformly along its path* a certain load, P lbs. say, is required in the box E. The effort causing sliding is the tension of the cord, which, omitting certain small corrections with which we need not concern ourselves, is P lbs. Our results will refer to the corrected value, *i.e.* the load in the box would not be exactly P lbs. but a little more, so that P lbs. is the pull of the cord on CD.

The ratio of P to W is called the **Coefficient of Friction**, and we shall see on what the value of it depends. Calling the coefficient f we have

$$P = fW.$$

First, keeping the surfaces in contact the same, how does P vary, when we alter W by altering the load in the sledge?

Result.—P varies directly as W, so that so far f is constant.

Next.—Is the force P altered by altering the area of the bearing surfaces, still keeping the same materials and load?

Result.—P does not alter.

The coefficient then is independent of the area of contact.

Next.—Does the speed of sliding affect f?

Result.—The sledge being set in motion at various speeds, P was the same for all. Therefore, *the coefficient is independent of the speed.*

Next.—Does P alter when we alter the condition of the bearing surfaces as regards smoothness or lubrication?

To this of course we should expect an affirmative answer, and this answer the experiments gave.

Thus — *The value of the coefficient depends on the nature of the surfaces.*

There is only one thing left to vary, viz. the material

of the surfaces; and here we must notice that it is difficult to say that two surfaces of different materials are in exactly the same condition, since condition cannot be measured, but only judged. It is evidently difficult to compare the smoothness, for example, of metals generally with that of wood; or even that of wrought-iron, with that of cast-iron, the grain being different.

Allowing for this, we find from experiment that the material also affects the value of P, which differs for different materials, even although each is finished in the best manner.

We combine all the foregoing results together in

The Ordinary Laws of Friction, viz.—The amount of friction between the elements of a sliding pair is equal to the total pressure between the surfaces multiplied by the coefficient. This coefficient being—

Dependent on the material and condition of the rubbing surfaces, but independent of the extent of the surfaces in contact and of the sliding velocity.

The experiments proving the above laws were carried out by Morin in 1831-33 in the manner we have roughly sketched, and they proved the truth of the laws *within the limits of pressure and velocity which he used*, viz. from 0 to 10 f.s., and from $\frac{3}{4}$ to 128 lbs. per square inch.

In much modern machinery these limits are far exceeded, and the methods of lubrication are so perfect, that the friction is in many cases that of a fluid between the surfaces of two solid bodies, but for the effect of these circumstances we must refer to more advanced treatises.

The experiments involved the measurement of P and W, and hence gave the values of f. Some of the more important of these values are to be found in the appended table.

Nature of Surfaces.	Condition of Surfaces.	Coefficient of Friction.
Wood on wood .	Perfectly dry and clean	·25 to ·5
Metal or wood on Metal or wood	Slightly oily . .	·15
Do. do. . .	Well lubricated .	·07 to ·08
Do. do. . .	Lubricant constantly renewed	·05

The coefficient of friction of a carriage, looked at as forming an element of a sliding pair, is generally called the *Draught*, and its value is usually expressed in lbs. per ton weight. It varies of course considerably with the state of the roads, so only mean values will be given. These may be taken at

55 lbs. on a macadamised road
20 lbs. on a stone tramway
8 lbs. on a railroad (very slow speeds).

In the case of a train, however, the air offers a great resistance which increases very rapidly with the speed, so that at 60 miles per hour the resistance is believed to exceed 50 lbs. per ton. The laws of this resistance are outside our limits.

Friction of Rest — Limiting Friction.—The friction we have dealt with is that existing between moving bodies, or the friction of motion. When, however, we consider the process of starting a body, a different law of frictional resistance exists. There is then, at any instant, as we gradually apply the effort, an amount of friction called into play just sufficient to balance the effort. As the effort increases so does the friction, until it reaches a certain limiting value, beyond which it cannot go; any further increase of effort then causes motion. This limiting value is called the Friction

of Rest or Limiting Friction, and *its* value follows the same laws as the Friction of Motion, with, however, a slightly greater coefficient in most cases. Its value is not of importance to us; more especially because any slight jar, during the starting, causes the body to start directly the effort is greater than the value found for the friction of motion, so it would never be safe to reckon on the friction of rest to prevent motion.

EXAMPLES.

1. The diameter of the piston of an indicator is $\frac{5}{8}$ in., the steam pressure under it is 30 lbs. absolute; the atmosphere presses on the top, and it is kept down by a spring which requires a force of 32 lbs. to compress it 1 inch. Find how much the spring is compressed. *Ans.* .147″.

2. The piston of a steam cylinder is 90 ins. diameter, the piston rod diameter is $8\frac{1}{2}$ ins., and there is no tail rod; the cylinder is horizontal. Find the effective effort of the steam—1st, when the pressure at the back of the piston is 16 lbs. absolute, and that in front $3\frac{1}{2}$ lbs. absolute; 2d, when these are reversed. *Ans.* 79746 and 78654 lbs.

3. The steam pressure in a boiler is 120 lbs. by gauge. One safety valve is $3\frac{1}{2}$ ins. diameter, and the spring keeping it in place is compressed $3\frac{1}{2}$ ins. from its original length. Find the increase of pressure necessary to lift the valve $\frac{1}{8}$ in., which is the ordinary lift allowed. Also if a stop be fitted which prevents the valve rising more than one fourth of its diameter. Find what pressure would force it up against the stop.
Ans. $4\frac{2}{7}$ and 150 lbs. per sq. in.

4. The diameter of a piston is 54 ins., stroke 3 ft. The piston approaches within 1 in. of the end of the cylinder when at the end of its stroke, and the steam is cut off at half stroke. The boiler pressure is 130 lbs. by gauge, and there is a drop of 10 per cent on the absolute pressure between the boiler and the cylinder. Find the pressure in the cylinder at each $\frac{1}{10}$ of the stroke, assuming the simple hyperbolic law of expansion.
Ans. 130.5 to half stroke, 109.7, 94.6, 83.2, 76.5, 67. lbs. per sq. in.

5. The spring ring in the preceding is 6 ins. wide, and the pressure between it and the cylinder is 3 lbs. per square inch.

Find the frictional resistance to motion, the surfaces being well lubricated. *Ans.* 214 lbs.

6. Draw by graphic construction a curve of effort for (4), and show how to represent on it the effect of (5) in reducing the effective effort.

Ans. The friction is equivalent to a loss of .1 lbs. per sq. in. on the piston, therefore the curve is lowered by 229 lbs.

7. A horse walking at 2 miles per hour can exert a pull of 166 lbs., and at 4 miles per hour a pull of 83 lbs. Find the total load he can move at those speeds on a road.

Ans. 3 tons and $1\frac{1}{2}$ ton.

CHAPTER III

WORK AND ENERGY

THE effect produced by the movement of a pair is the overcoming of a resistance through a certain distance, and it is for the production of this effect that the pair is, in the great majority of cases, required. This effect is spoken of as doing **Work**, and we say *work is done against the resistance.*

The general meaning of the word Work is of course well understood; but it is not in this general sense we use it in Mechanics, but strictly in the limited sense defined above.

Let us consider a simple case of sliding, viz. the raising of a weight, then the weight slides relatively to the earth against the resistance of gravity. Work then is done against gravity.

Now let us consider how the motion is caused; some effort is required, and we may take this to be the muscular effort of a man. The man then exerts an effort through the distance the weight is raised. In ordinary language we say "the man is doing work," but in Mechanics we say he is **exerting energy**, which causes work to be done against the resistance.

The action of lifting the weight or generally of moving a sliding pair against a resistance has then two descriptions; according, we may say, to the point of view from which we regard it. Looked at from the effort side it is

called *exerting energy*, from the resistance side, *doing work*.

In order that energy be exerted, or work done, we must have the combination of force and motion. For example, no work is done when the man stands simply holding the weight, although it may be his duty to do so, and he would in ordinary language undoubtedly be working since he would suffer fatigue; but there is no motion, and hence, according to our definition, no work done, or energy exerted. Again, in the sliding of a weight along a perfectly smooth horizontal table, if such a thing could be, no work would be done, because there would be no resistance, although there is motion.

In all cases of doing work, we shall find there are at least three bodies to be considered.

One supplies the effort causing the motion, and thus exerting energy—*e.g.* the man lifting the weight.

One is moved, by the effort, against the resistance—*e.g.* the weight itself.

The third resists the motion, being the source of resistance—*e.g.* the earth, which is the source of the gravitation resistance.

In the case we have considered, the first and third are natural sources of effort and resistance, but we shall see as we advance that this is not necessary; but what is necessary is, that the first be connected in some way to a natural source of effort, and the third to one of resistance.

We have defined *doing work, or working*, but we cannot give any particular definition of the term Work by itself, but of Energy this is not the case.

A source of effort in nature can exert energy, and we say it possesses energy; now what it does possess is the power of causing work to be done, so that by using the term energy we make it mean *power of causing work to be done;* and this therefore is the definition of energy. The source then exerts energy when it puts forth this power.

Conservation of Energy.—There is now a law of nature which tells us that energy can never be destroyed or lost; it can be transformed, can be transferred from one body to another; but no matter what transformations are undergone, when the total effects of the exertion of a given amount of energy are summed up, the result will be exactly equal to the amount originally expended from the source. This law is called the Conservation of Energy, and it will be our task to apply it and trace its action in cases commencing with the simplest and going on to others of a more complex nature, and we must first consider how to measure amounts of *energy exerted* or of *work done.*

Unit of Work.—Both the quantities just mentioned being, as we have seen, only different views of the same actions, and each consisting of the same components, viz. force and distance—will accordingly be measured by the same unit, which we may therefore call the Unit of Energy or of Work.

The simplest kind of work we are acquainted with is the lifting of a weight, and we are thus led to select from this our unit. The unit weight being 1 lb. and unit force 1 ft., *the work done in lifting* 1 *lb. through* 1 *ft.* is taken as the **Unit of Work,** and is called 1 foot-pound or 1 ft.-lb.

We may, of course, if we please, take other units, say 1 ton lifted through 1 inch; which we call 1 inch-ton, and so on. But in all cases the idea involved is that of lifting.

We have defined the unit as the work done in lifting, but we might equally well define it as the *energy exerted to cause the lift of* 1 *lb. through* 1 *ft.*, and thus it is the *unit also of energy.*

The idea of distance is the same in each way of looking at the action; but the difference is, that the 1 lb. is in the one case resistance, and in the other effort; but these are in this case exactly equal, each being the 1 lb.

We have so far used the idea of lifting. But now we know that it will take exactly the same amount of energy to push a piece through 1 ft. in any direction against a direct resistance of 1 lb. as to lift it against the pull of the earth of 1 lb., in fact the last statement is included in the preceding one. We can then, if we please, dissociate the unit from the idea of lifting, and define it simply as *the work done, or energy exerted in overcoming a resistance of* 1 *lb. through* 1 *ft.*

It is quite immaterial which way we state the definition. But we must clearly realise that the direction of motion is of no consequence, so long as the resistance is directly opposed to the motion. The importance of this last qualification we see in the next chapter.

We can then now measure the energy exerted or work done in moving a sliding piece through a given distance against any given resistance. For let

$$R = \text{resistance in lbs.}$$
$$s = \text{distance moved in ft.}$$

Then

$$\text{Work done} = R \times s \text{ ft. lbs.} \qquad (1).$$

Or equally

$$R = \text{effort in lbs.,}$$

and

$$\text{Energy exerted} = R \times s \text{ ft. lbs.} \qquad (2).$$

Whence we see that in this case

$$\textbf{Energy exerted} = \textbf{work done} \qquad (3).$$

We have here our first example of the—

Principle of Work—Balanced Forces.—The equation (3) we have derived from the equality of effort and resistance. These we have taken to be equal, which we can also express by saying that the forces acting on the piece balance, each being R lbs., and thus we call this a case of Balanced Forces.

The balance is in this case plain, effort and resistance being directly opposite and equal in amount. But

it will not in all cases be so evident, and we may have a balance without such equality of effort and resistance in lbs. as we have here (see turning pair, page 64). To all such cases, however, the principle of work, in its simplest of all forms as just given, will apply. And we are not to look upon this statement as derived from other principles, but as in itself a first principle, which we can in any instance verify, but which must not be taken to require proof.

The principle of work is only a statement, for this case, of the conservation of energy, and we will now see how that doctrine reduces to its present simple form.

The case to which we say it applies is that of Balanced Forces. Now, if the forces are balanced, there is no excess left to produce changes in the body moved. The body moved is the agent by which the energy causes the doing of the work, and we can also then group all these cases under the heading of—No change produced in the agent used. Using this definition, there is no effect produced by the energy exerted except the doing of work. Hence by conservation of energy

$$\begin{aligned}\text{Energy exerted} &= \text{sum of all effects produced,}\\ &= \text{work done only.}\end{aligned}$$

For example:—A railway truck weighs 2 tons, resistance 20 lbs. per ton. Find the energy exerted in keeping it moving at a constant speed through 50 yds.

This is a case of balanced forces, and we have

$$\begin{aligned}\text{Energy exerted} &= \text{effort} \times \text{distance,}\\ &= 40 \text{ lbs.} \times 150 \text{ ft.,}\\ &= 6000 \text{ ft. lbs.}\end{aligned}$$

Had the question been set us *to move* the truck 50 yds. instead of "keeping it moving at a constant speed," we should not have a case of balanced forces, because the process would involve starting and stopping, during which processes, as we shall see hereafter, the forces would not be balanced.

But the larger statement of "no effect on the agent" would hold good, because the state of the truck would have been on the whole unaltered, and the principle of work in its simplest form applies. There is, on the whole, a balance, but not at each instant. This we shall not now inquire into, but only note it to show that the simple statement of the principle applies to such cases, *i.e.* motion from rest to rest again.

In the two cases of sliding we have so far considered, one body has, in each case, been the earth; but this is not at all necessary. For in any case whatever

$$\text{Energy exerted, or work done,} = \text{Resistance to relative motion} \times \text{distance moved relatively.}$$

For example:—To plane a piece of material we may use a machine, in which the piece is fastened to a moving bed, the tool being still (Fig. 43); or one in which the tool moves, the work being still (Fig. 44).

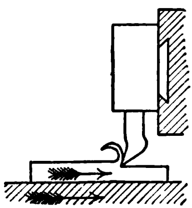


Fig. 43.

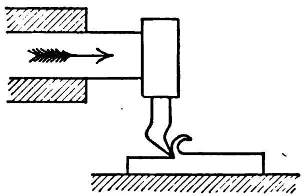


Fig. 44.

If now R be the resistance offered by the metal to the point of the tool, and l be the length of the work, in lbs. and ft. respectively. Then evidently, in each case,

$$\text{Work done in one cut} = R \times l \text{ ft.-lbs.},$$

quite irrespective of which moves. But there is this difference, that the energy is exerted in opposite manners in the two cases. In the first case the effort moves the work against the resistance of the tool. In the second the effort moves the tool against the resistance of the

work. Both require the same amount of energy, but it must be applied to a different part of the machine.

["The work" above means the piece of material, that being the usual workshop term.]

In addition to the examples we have already considered, the principal examples of sliding pairs, in which energy and work require calculation, are engine pistons and pump plungers.

With regard to the first, all we need notice is that it is immaterial whether the cylinder be still or whether it oscillate, the calculation of energy is unaltered.

In pumps the work to be done is the lifting of water, and we can simplify the calculation in such cases by the following methods :—

Lifting of a Number of Weights.—Let W_1, W_2, etc., be the weights, and let them be initially at heights Y_1, Y_2, etc., above a given datum level.

Let now W_1 be lifted a distance y_1, W_2 a distance y_2, etc.

Their new heights are then Y_1+y_1, Y_2+y_2, etc., which we will call Y'_1, Y'_2, etc. Then

$$\begin{aligned}\text{Work done} &= W_1y_1+W_2y_2+\ldots,\\ &= W_1(Y'_1-Y_1)+W_2(Y'_2-Y_2)+\ldots,\\ &= \{W_1Y'_1+W_2Y'_2+\ldots\}-\{W_1Y_1+W_2Y_2+\ldots\}.\end{aligned}$$

Now let

$$\overline{Y}=\text{height of C. G. originally,}$$
$$\overline{Y'}=\text{height of C. G. finally.}$$

Then

$$\overline{Y'}=\frac{W_1Y'_1+W_2Y'_2+\ldots}{W_1+W_2+\ldots},$$

$$\overline{Y}=\frac{W_1Y_1+W_2Y_2+\ldots}{W_1+W_2+\ldots},$$

$$\begin{aligned}\therefore\ \text{Work done} &= (W_1+W_2+\ldots)(\overline{Y'}-\overline{Y}),\\ &= \text{total weight}\times\text{lift of C. G.}\end{aligned}$$

In the case now of a shaft full of water, we need not

consider the separate lifts of the pump, but only the total lift of the C. G. of the whole body of fluid; or equally, in calculating the energy which the water stored in an elevated reservoir could exert by descending.

Motion in a Curve—Turning.—It is not necessary that the relative motion of two pieces be rectilinear sliding for the preceding calculations to apply, *so long as the resistance is directly opposed to the motion.* For consider such a motion as a piece along a curve AB, the paper representing the other piece. Then, by dividing the curve up into a large number of small portions as *ab*, of length x, we have a number of rectilinear slidings each giving

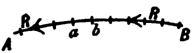


Fig. 45.

$$\text{Work done} = Rx,$$

and finally, for the whole,

$$\begin{aligned}\text{Work done} &= R \times \text{sum of } x\text{'s},\\ &= R \times \text{curved length AB},\end{aligned}$$

but the italics above must be carefully attended to.

We will now consider the turning pair balanced.

The turning pair consists of a shaft, turning in a bearing, and having arms to which efforts and resistances can be applied.

We must first determine the necessary conditions for balance. If there be only one arm, as in Fig. 46, and the effort and resistance be applied to it exactly opposite to each other, then evidently $P = R$, *i.e.*

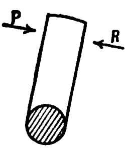


Fig. 46.

$$\text{Effort} = \text{resistance}.$$

But if, as is generally the case, the resistance be not applied to the same arm, or if to the same arm not directly opposite, we must have some different condition. In Fig. 47 P and R are applied to arms of different lengths.

Now we can determine the condition of balance, either from Statics or by the principle of work. We will select the latter method.

Let the crank move to the new position as shown,

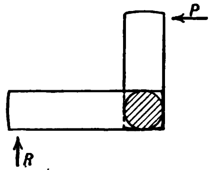


Fig. 47.

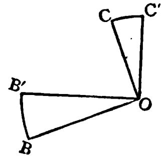


Fig. 48.

B to B′, C to C′ (Fig. 48), the centre lines only being shown, then

$$\text{Energy exerted} = \text{P} \times \text{arc BB}',$$
$$\text{Work done} = \text{R} \times \text{arc CC}',$$

whence

$$\text{P} \times \text{arc BB}' = \text{R} \times \text{arc CC}' \qquad (1).$$

But

$$\text{BB}' = \text{OB} \times \text{angle BOB}' \quad \text{(circular measure)},$$

and

$$\text{CC}' = \text{OC} \times \text{angle COC}'.$$

Also

$$\text{Angle BOB}' = \text{angle COC}'.$$

Whence, from (1),

$$\text{P} \times \text{OB} = \text{R} \times \text{OC}.$$

This then is the condition of balance, or in words

Moment of effort = moment of resistance.

But this is the principle we know in Statics as the Principle of the Lever, and we have thus proved this principle by means of the Principle of Work. Or we may say, if we please, that we have verified the Principle of Work.

We have in Fig. 47 drawn both P and R at right angles to the cranks on which they act. This was for simplicity, but we shall now see that there is no necessity for this to be the case.

For example, let the turning body, outside the bearing,

be of any irregular shape (Fig. 49), turning about O the centre of the bearing.

Further, let P and R be applied at A and B, but not at right angles to OA and OB; keeping, however, always at the same angles to OA and OB during the motion.

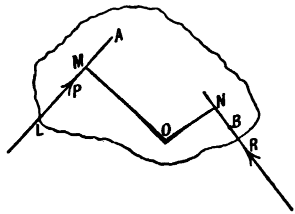


Fig. 49.

Draw OM and ON perpendicular respectively to the directions of P and R.

Then because, during the motion, P will always act along the line AM in the moving body; P may just as well be applied to M as to A.

For example, let it be applied by a rod LA, then LA always lies along MA, and we may as well fasten the part MA of the rod to the body. The remaining part then pushes at M.

Similarly R's effect is exactly the same as if it were applied at N instead of B.

Moments.—The condition of balance then will be

$$P \times OM = R \times ON,$$

i.e.

Moment of effort = moment of resistance.

It appears then that we do not need to know either the effort or resistance in pounds, but only their moments; a force of 1 lb. acting on an arm of 2 ft., for example, being identical in its effect on the motion of a turning pair, with a force of 2 lbs. at an arm of 1 ft., or 4 lbs. at 6 inches, and so on.

Since a moment consists of force multiplied by distance, it must be measured in terms of the product of feet and lbs., or of inches and tons, etc. But work is measured in foot-lbs. or inch-tons; and so, in order not to confuse the essentially different magnitudes of moment and work, we will express moment, not as ft.-lbs., etc.,

etc., but as lbs.-feet, tons-inches, etc., inverting the order.

Moment, we may say, bears the same relation to a turning pair that a pull or push does to a sliding pair, since by moment only can the motion of a turning pair be caused or be resisted. Now, in the case we have taken of a shaft with an arm to which P the effort is applied, the effect of P is not simply to produce the turning motion, but also to produce a pressure between the shaft and its bearing; let us see, then, what is necessary simply to produce turning and nothing else.

The answer to this is that we must have two arms equal and opposite, to each of which we apply an effort P, Fig. 50. Then there will be no effect produced between the shaft and bearing but pure turning.

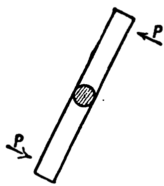


Fig. 50.

We act naturally in this way in many cases. For example, to turn a person round, we should take hold say of one arm in each hand, and then push with one hand and pull with the other; but if we only took hold of one side and pulled say, we should not only turn, but also pull him towards us. Or in using a rimer to enlarge a circular hole, if we wish to make the hole true we use a double

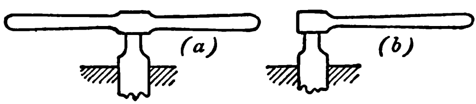


Fig. 51.

handle, Fig. 51 (*a*), because if we used a single one, Fig. 51 (*b*), we should have to take hold of the rimer with the other hand to hold it upright. If not it would topple over when we pushed the end of the handle, if not entered far in the hole; and although if it were entered some distance in the hole it would not topple over, the

effect would still appear in the untruthfulness of the hole.

Turning then can be caused by a pair of forces as Fig. 50. Such a pair we call a couple, and their only effect is a turning moment. A single effort produces a turning moment and also a pressure on the bearing, but the latter we do not at present care about, because it has not directly any effect on the turning. It has an indirect effect by producing friction, which we shall discuss later.

The calculation of energy and work then leads us to the problem—To find the work done in moving a turning pair through a given angle against a given resisting moment.

Turning back to page 64 we have (Fig. 48),

$$\text{Energy exerted} = \text{P} \times \text{arc BB}',$$
$$= \text{P} \times \text{AB} \times \text{angle BAB}'.$$

But

$$\text{P} \times \text{AB} = \text{moment of effort},$$
$$\text{BAB}' = \text{angle turned through},$$

and hence

$$\text{Energy exerted} = \text{moment of effort} \times \text{angle turned through},$$

or equally

$$\text{Work done} = \text{resisting moment} \times \text{angle turned through}.$$

Comparing with the sliding pair (page 59) we see that moment takes the place of force, and angle that of distance.

Wasted Work.—In the motion of a sliding or turning pair it is usually the case that the resistance which balances the effort is not one but two forces, viz. the useful resistance, and the friction of the pair.

For example, take the piston of a steam engine in a horizontal cylinder. Then if the weight be W, the forces balancing the effort of the steam on the piston are two, viz. the resistance of the piston rod and the friction of

the piston in the cylinder. This friction depends on two things, first on W; the piston resting on the solid piece

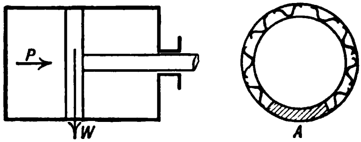


Fig. 52.

A, this in turn bearing on the spring ring, and this again on the cylinder bottom; due to W we shall have friction of amount fW. Also we have the ring pressed out against the cylinder by the springs, and if their total pressure be S, we have additional friction fS due to this cause. In each f is the coefficient of friction between the ring and cylinder. If then

$$P = \text{effort of steam},$$
$$Q = \text{resistance offered by piston rod, or useful resistance.}$$

We have

$$P = R = Q + fW + fS,$$

R representing total resistance. If

$$s = \text{stroke},$$
$$\text{Work done per stroke} = Rs = Qs + f(W + S)s.$$

Now these two terms, Qs and $f(W+S)s$, are essentially different in their nature. Movement of the piston rod is what we wanted to produce, and Qs is then the useful effect. But the second term we do not want at all, it is inseparable from working, yet we try always to decrease it as much as possible; and this we do because it produces no useful effect. We know what it does do, viz. heat up the bearing surfaces, but this is an effect we would rather if possible be without. The Principle of Energy forbids us to say that energy or work is lost; but since the energy exerted in overcoming the friction is changed into a form, viz. Heat, in which we cannot utilise it, we say that amount is wasted, and we call it Work Wasted.

We get then now a second form of the **Principle of Work** for **Balanced Forces**, viz.—

Energy exerted = useful work done + work wasted,

or we generally omit useful and say simply

Energy exerted = work done + work wasted.

The calculation of work wasted during the motion of a sliding pair we have seen, therefore if

S = total pressure between the pieces,
fS = frictional resistance,

and

Wasted work = fS × space moved relatively.

Axle Friction.—For the turning pair we have not yet seen how the friction is reckoned, this we will now consider.

We have seen that Moment is the form taken by resistance to turning motion, and we express then the frictional resistance as a Moment.

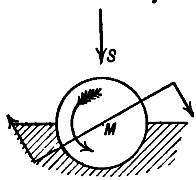


Fig. 53.

The friction of axles, as turning friction is generally called, can be connected with sliding friction, and the ordinary law is taken as the same, very heavy pressures and velocities not being here considered, and, as before, the lubrication being supposed imperfect.

Let

S = the resultant pressure forcing the turning piece into its bearing.

Then the rubbing friction produces a resisting movement M given by

$$M = f'S \cdot r, \quad r = \text{radius of bearing.}$$

The coefficient f' we take to follow the same *laws* as f for sliding, but it has not the same value, being determined from independent experiments on axles. An

average value under ordinary conditions for metal bearings is from $\frac{1}{16}$ to $\frac{1}{18}$.

We can now at once determine the work wasted in turning through any angle; generally we reckon per revolution. Thus

$$\text{Work wasted per revolution} = \text{friction moment} \times \text{angle turned},$$
$$= f'Sr \times 2\pi = \pi f'Sd,$$

where

$$d = \text{diameter of bearing.}$$

Graphic Representation.—Graphic construction enables us to represent very simply quantities of energy or work. Let

$$P = \text{effort in lbs.}$$
$$s = \text{space moved in feet.}$$

On a scale of x inches to 1 foot make AB to represent s ft. AB is then sx inches. x may be a whole number

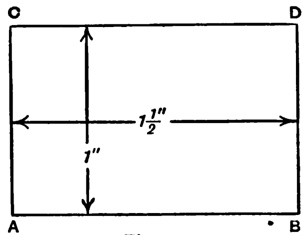


Fig. 54.

or a fraction. On a scale of y inches to 1 lb. make AC to represent P lbs. AC is then Py inches.

Complete the rectangle AD. Then

$$\text{Area of AD} = \text{AC} \times \text{AB},$$
$$= Py \times sx \text{ sq. ins.},$$
$$= Ps \times xy \text{ sq. ins.}$$

Thus the area AD represents Ps ft.-lbs. on a scale of xy sq. ins. to 1 ft.-lb. AD then represents the energy exerted by P, on a scale compounded of the scales of length and force originally chosen. For example:—

Let P be 2000 lbs. pressure on a piston, s the stroke being 3 ft.

Then the scales would be say:

$$\tfrac{1}{2} \text{ in. to 1 ft.} \quad \text{Thus } AB = \tfrac{1}{2} \times 3 = 1\tfrac{1}{2} \text{ in.}$$

and

$$\tfrac{1}{2000} \text{ in. to 1 lb.} \quad \text{Thus } AC = 2000 \times \tfrac{1}{2000} = 1 \text{ in.}$$

These are the dimensions of Fig. 54. The compound scale is then

$$\tfrac{1}{2} \times \tfrac{1}{2000} \text{ sq. in. to 1 ft.-lb.}$$

Then

$$\begin{aligned} \text{Area } AD &= 1\tfrac{1}{2} \text{ sq. in.,} \\ &= 6000 \text{ ft.-lbs.} \end{aligned}$$

which is the energy exerted.

In this method the path of s ft. may be either straight or curved. If it be straight AB is an actual representation of it, but if curved then AB represents it straightened out.

We can in this manner represent the work done in a turning pair. For let

$$\begin{aligned} R &= \text{resistance,} \\ r &= \text{radius at which R acts.} \end{aligned}$$

Then for one revolution, we make AB to represent $2\pi r$ ft., and AC to represent R lbs.

But now the work done is also given by the moment Rr overcome through 2π, and we shall see that exactly

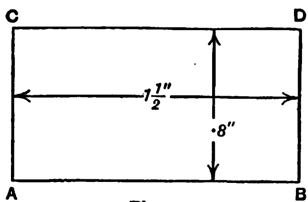


Fig. 55.

the same diagram will also represent the energy or work when reckoned in this way. For let

$$R = 80 \text{ lbs.}, \quad r = 3 \text{ ft.}$$

Then draw AB on a scale of 1 in. to 4π ft. or $1/4\pi$ in. to 1 ft. to represent $2\pi r$, *i.e.* 6π ft.

$$\therefore \text{ AB} = 1\tfrac{1}{2} \text{ in.}$$

and AC on a scale of $\frac{1}{100}$ in. to 1 lb., to represent R.

$$\therefore \text{ AC} = \tfrac{80}{100} = \tfrac{4}{5} \text{ in.}$$

Then

$$\text{Work done} = \text{area AD} = \tfrac{3}{2} \times \tfrac{4}{5} = \tfrac{6}{5} \text{ sq. in.}$$

And the scale is

$$\frac{1}{4\pi} \times \frac{1}{100} \text{ sq. in. to 1 ft.-lb.}$$

$$\therefore \text{ Work done} = \tfrac{6}{5} \times 4\pi \times 100 = 480\pi \text{ ft.-lbs.}$$

Next Method.—Resisting moment $= Rr = 240$ lbs.-ft., and AC above will represent this on a scale of $\frac{4}{5}$ in. to 240 lbs.-ft., *i.e.* $\frac{1}{300}$ in. to 1 lb.-ft. Also

$$\text{Angle turned} = 2\pi,$$

and AB will represent this on a scale of $1\frac{1}{2}$ in. to 2π, *i.e.* $3/4\pi$ in. to unit angle. And then

$$\begin{aligned} \text{Work done} &= Rr \times 2\pi, \\ &= \text{AC} \times \text{AB} = \text{area AD}, \end{aligned}$$

as before.

And for the scale we have $\frac{1}{300}$ in. $\times$ $3/4\pi$ in. to 1 lb.-ft. $\times$ unit angle, *i.e.* $1/400\pi$ sq. in. to 1 ft.-lb. of work, which is exactly the scale on which we measured AD before; which of course should be, as the result must be the same.

The same diagram then which we use for a sliding pair represents work done in a turning pair; only we use a scale of moment and a scale of angle, instead of a scale of force and a scale of distance. The compounded scale gives in each case the scale of energy, or of work done.

There is no gain in such simple cases as we have just taken in graphic representation, but the method is

best understood in simple cases, and the gain will be seen when we come to more complicated ones.

Power—Horse Power.—The definitions of Energy and of Work done contain no reference to the speed at which the operations considered have to be carried on; thus to lift one lb. through one ft. requires the same amount of energy to be exerted whether it be done in one second, or one minute, or one hour.

[The speed of a certain operation has an effect generally on the energy required, because it affects the values of the useless resistances, *i.e.* those which waste energy; but this does not affect what is stated above where we take the resistance as given.]

But to compare the value of different working agents, we must consider *the energy they can respectively exert in a given time.* This quality of an engine, or other energy exerting agent, we call its **Power.** The Power of an engine, for example, then means the energy it can exert in a given time.

To measure power we require some **Unit Power,** and this we take as the capacity for exerting 33,000 ft.-lbs. in one minute. The unit is called a **Horse Power,** and is due to Watt; being introduced by him, so that purchasers of engines should be able to compare their values with those of horses. The unit is much above the power of any horse in continuous work; but the extra margin was allowed to prevent any possibility of a mistake in the other direction.

The Horse Power then of any agent is equal to the number of ft.-lbs. of energy it can exert in a minute divided by 33,000.

If, for example, a steam engine work with constant steam pressure p lb. per sq. in., piston area A sq. ins., stroke s ft., revolutions per minute N. The calculation of the Horse Power proceeds thus—

$$\text{Effort} = p \times A = pA \text{ lbs.}$$
$$\text{Distance moved in one revolution} = 2s \text{ ft.}$$

$\therefore$ Energy exerted in one revolution $= pA \times 2s$ ft.-lbs.

$\therefore$,, ,, minute $= pA \times 2s \times N$ ft.-lbs.

$$\therefore \text{ Horse power} = \frac{pA \times 2s \times N}{33{,}000}.$$

and similarly for any other agent.

Examples.

1. A man weighing 150 lbs. carries loads of 144 lbs. to a height of 30 ft. Find the work done in each journey. If the man exerted the same amount of energy in lifting the weights by means of a winch, which wasted one-third of the energy applied, how much more useful work could he do?
Ans. 8820 ft.-lbs.; .3 times.

2. Iron pigs six inches square lying originally on the ground, are built into a stack 6 ft. by 5 ft. by 4 ft. high. Find the work done. *Ans.* 90720 ft.-lbs.

3. A bicyclist and machine weigh 180 lbs. Find the H. P. he exerts when riding at 20 miles per hour, on a track the resistance of which is 1 per cent of the weight. *Ans.* .096.

4. A colliery engine raises loads weighing 21 cwt. from a depth of 1200 ft. in 45 seconds. Find the H. P. required.
Ans. 114.

5. Compute the nett H. P. required to pump out a basin with vertical sides in 48 hours, the area of the water surface being 50,000 sq. yds., and depth of water 20 ft., the water being delivered at a height of 26 ft. above the bottom of the basin. (35 c. ft. of sea water weigh 1 ton.) *Ans.* 97.

6. A boiler 12 ft. diameter, 10 ft. long, full of water, is to be pumped out, the water being delivered at the sea level. The ship's draught is 21 ft., and the bottom of the boiler is 4 ft. above the keel. Find how long two men each capable of exerting 2500 ft.-lbs. per minute would take, using a hand pump in which $\frac{8}{10}$ of the energy is wasted. *Ans.* 3 hrs. 28 min.

7. A pumping engine 9 ins. diameter, 6 ins. stroke, making 80 revolutions per minute, is fitted in the above ship. Assuming a constant steam pressure (effective) of 40 lbs. per sq. in.; find how long the engine would take in pumping out the boiler, assuming that engine and pump together waste one half the energy.
Ans. 8 minutes.

8. The resistance of a ship at 15 knots is 70,000 lbs. Of the energy exerted by the steam 15 per cent is wasted before reaching the crank shaft, and 50 per cent altogether. Find the H. P. of the engines. There are two propellers, and the revolutions are 80 per minute. Find the moment exerted on each shaft.

Ans. 6430, 960 tons-ins.

9. Each engine in the preceding has three cylinders, the work being equally divided between them. The stroke is 3 ft. 6 ins., and the diameters are 70 ins., 48 ins., and 32 ins. Find the effective steam pressure in each cylinder.

Ans. 7.73, 16.44, 36 lbs. per sq. in.

10. An anchor weighs $5\frac{3}{4}$ tons, and the cable 1 ton per 8 fathoms. The ship is anchored in 9 fathoms, and the deck is 10 ft. above the water line. The cable hangs in the quadrant of a circle, touching the ground where the anchor lies. Find the work done,—1st, while hauling in the slack; 2d, while lifting the anchor. Neglect the buoyancy of the water. Distance of the C. G. of the arc of a quadrant from either bounding radius $=\frac{2}{\pi}$ times the radius. *Ans.* 33.58, 474.66 ft.-tons.

11. A water-power engine of 10 H. P. is supplied from a tank 12 ft. by 8 ft. by 6 ft., at a height of 120 ft. Supposing the tank be full but no supply, find how long the engine could run.

Ans. 13 minutes.

12. A locomotive weighing 30 tons draws a train weighing 65 tons at 50 miles per hour, resistance 33 lbs. per ton. Find the H. P. required. If the train became detached, the engine still exerting the same power, what speed would it attain, assuming the resistance to vary as the speed squared.

Ans. 418, 73.5 miles per hour.

13. In question 8, assume the resistance to vary as the cube of the speed, and find the H. P. required to propel the ship at 18 knots. *Ans.* 13,300.

14. A crank shaft, diameter $12\frac{1}{2}$ ins., weighs 12 tons, and is also pressed against the bearings by 36 tons horizontal. Find the H. P. lost in friction at 90 revolutions. Coefficient .06.

Ans. 45.6.

CHAPTER IV

OBLIQUE AND VARIABLE FORCES—INDICATOR DIAGRAMS

IN the preceding chapter the efforts and resistances considered have been constant, and in the line of motion. We now proceed to consider more general cases.

Oblique Action.—In the motion of a sliding pair as A and B (Fig. 56), the resistance may not be directly against the motion, but inclined, as R at an angle θ.

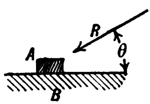


Fig. 56.

To find the effect of this we resolve R, by the laws of Statics, into two components, R cos θ and R sin θ, then the effect of R is the same as the effect of two separate forces, as in Fig. 57.

In considering these forces separately, we see first that the magnitude of R sin θ can, in the absence of friction, have no effect on the effort required to move A along B. We can never of course practically get rid of the effect of R sin θ, but we can, by successive improvements in smoothness of surface and in lubrication, continually diminish it; and the only obstacle to complete elimination of its effect is imperfection of surface and of lubrication. We can thus conceive that, *in itself*, it is incapable of offering any resistance at all to the motion, but can only do so by the friction it can excite.

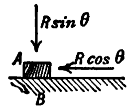


Fig. 57.

Neglecting friction then

$$\text{Effort} = \text{R} \cos \theta,$$

and for a movement s,

$$\text{Energy exerted} = \text{R} \cos \theta \times s,$$
$$= \text{Resolved resistance} \times \text{distance moved.}$$

We can express this in a slightly different manner, which will be found useful.

For let ab be the movement (Fig. 58). From b let fall bc perpendicular to R's direction, then

$$\text{Energy exerted or work done} = \text{R} \cos \theta \times s = \text{R} \times s \cos \theta,$$
$$= \text{R} \times ac,$$
$$= \text{Resistance} \times \text{resolved distance},$$

because the motion ab consists of a motion ac directly against R, and a motion cb at right angles to R.

Thus we can either resolve the resistance in the direction of the motion, or the motion in the direction of the resistance.

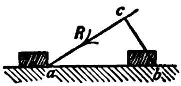


Fig. 58.

We can now apply this to the case in which motion takes place in any curve, against a resistance constant in direction. Such a resistance would be, for example, the pull of the earth on a weight.

Let a weight, W lbs., be lifted along the path shown from A to B.

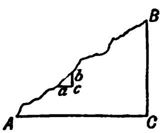


Fig. 59.

The constant resistance is W lbs. vertically.

Draw now BC vertical and AC horizontal.

Then, whatever be the shape of AB, the total motion is composed simply of AC and CB, and from the preceding we say

$$\text{Energy exerted or work done} = \text{W} \times \text{BC}.$$

We have not strictly followed the preceding because *ab* there was a straight line; but we can do that by dividing AB into a large number of small parts as *ab*, each of which is straight if we take them small enough.

Then for *ab*

$$\text{Energy or work} = \text{W} \times bc,$$
$$\therefore \text{Total energy or work} = \text{W} \times \text{sum of all the small heights,}$$
$$= \text{W} \times \text{BC}.$$

We are now able to examine in another way the case of turning, which we considered in the last chapter, viz. when the resistance or effort is not at right angles to the radius.

The figure (Fig. 49) on page 65 is here reproduced (Fig. 60).

Now resolve P along, and at right angles to, the radius OA. Then the part along OA is always at right angles to the motion, this being *at the instant* along the tangent to the circle in which A moves. Therefore, as we have seen, it can exert no energy, and therefore the Effort is simply P sin OAM at right angles to OA.

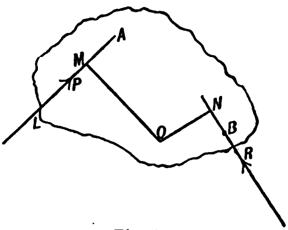


Fig. 60.

$$\therefore \text{Turning moment} = \text{P} \sin \text{OAM} \times \text{OA},$$

which is of course identical with P × OM, the result obtained on page 65.

The part P cos OAM will, of course, practically affect the motion, because it will pull the shaft against its bearing, and thus create friction. So actually it becomes a source of, not effort, but resistance. Whether there be friction or no then, the Effort will be only P cos OAM.

Similarly we can treat the resistance R, and obtain

as *the resistance against which work is done*, the force R cos OBN.

Variable Forces.—In all practical cases, the forces are not constant, but they vary. We require then to calculate the Work done against a varying resistance, or Energy exerted by a varying effort.

Let now AB represent the path of one piece of a sliding pair against a direct resistance.

Fig. 61.

Divide AB into a number of parts as shown, A*a*, *ab*, . . ., *l*B, and let the resistance given be constant during each small movement; its values being R_1 during A*a*, R_2 during *ab*, etc., to R_{11} during *l*B. Then

$$\text{Work done} = R_1 \times Aa + R_2 \times ab + \ . \ . \ . \ + R_{11} \times lB.$$

Now we will apply the graphic method of the preceding chapter.

To do this set up ordinates on A*a*, *ab*, etc., to represent R_1, R_2 etc.

We thus obtain a number of rectangles C*a*, etc., to *l*D, each representing the work done during the movement represented by its base. Therefore

$$\text{Total work done} = \text{sum of areas of rectangles} \qquad (1).$$

The tops of the rectangles constitute a stepped curve CD, and this curve has the property that—*at any point of the movement its ordinate represents the value of the resistance.* This we know by the manner in which we constructed it. Hence then we call CD the **Curve of Resistance.**

In the above the variations of R occur at definite

intervals. But the magnitude of the intervals does not affect the truth of the result (1). And hence it follows that the variations may follow each other as quickly as we please.

We shall still have

Work done = area between CD, the base AB, and the end ordinates. (2).

[We have here written the result (1) in different words, but it is a statement of the same fact.]

But when the variations follow one another with great rapidity, the steps of the curve CD become indefinitely short, and the curve becomes continuous as Fig. 62, still keeping the property stated in the italics above.

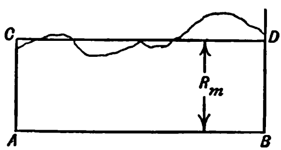


Fig. 62.

We have now then come to a case of continuously varying resistance, and we still have as before,

Work done = area of curve of resistance between end ordinates and base.

The case of varying moment is treated quite similarly, the base now representing angle turned through, and the ordinate representing the moment corresponding to the given angular position.

Mean Resistance.—We can, by the graphic method, determine the mean resistance during any given motion.

In chap. ii., page 22, the general sense of mean value was explained; and from it we see that by Mean Resistance or Mean Effort we understand that resistance or effort which, acting uniformly through the same distance, would produce the same effect, *i.e.* absorb the same amount of work, or exert the same amount of energy as is actually produced by the varying resistance or effort.

If then, in Fig. 62, we draw a rectangle whose area = area of curve, the height of this rectangle gives the mean resistance, or by calculation

$$R_{mean} \times \text{space moved} = \text{work actually done},$$
$$\therefore R_{mean} = \frac{\text{work done}}{\text{space moved}}.$$

We use whichever of the two methods, graphic or calculation, is simpler.

We will illustrate the methods by investigating some cases of varying force.

The Elastic Spring.—In Fig. 63, AB the broken line, represents the length l of an elastic spring. It is compressed to the length A′B by an effort applied to the end A. To find the work done in compressing it, set up A′C equal to R, the *final* value of the resistance, and join AC.

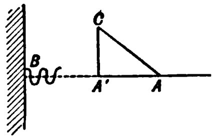


Fig. 63.

Then from page 47 the resistance at any point of the motion is represented by the ordinate of AC; so that AC is the curve of resistance. Hence

$$\text{Work done} = \text{area AA'C},$$
$$= \frac{R \times AA'}{2},$$
$$= \tfrac{1}{2} \text{ final resistance} \times \text{compression}.$$

An exactly similar value holds for extension.

The height of a rectangle equal to AA′C is evidently A′C/2, *i.e.* R/2,

$$\therefore \frac{R}{2} \text{ is the mean resistance.}$$

It is important to notice that the work done in compressing the spring is independent of the nature of the effort by which the alteration of length is effected.

For again from chap. ii.

$$R = P_0 \cdot \frac{AA'}{l} \text{ (page 47)},$$

$$\therefore \text{ Work done} = P_0 \cdot \frac{AA'}{l} \times \frac{AA'}{2},$$

$$= \frac{P_0}{2l} . AA'^2.$$

Now P_0 depends only on the size and material of the spring (see page 47), and not at all on the force acting on it. Hence our statement is proved. It matters not, then, whether the Effort be such as to exactly balance the Resistance at every point or no. If on the whole it just compress the spring to A′ and *have no other effect*, then

$$\text{Energy exerted} = \text{work done.}$$
$$= \frac{R \times AA'}{2},$$

and

$$\text{Mean effort} = \text{mean resistance,}$$

each being R/2 (compare page 60). The Resistance here dealt with is reversible, and the spring can give out again by expanding an amount of energy equal to the work done on it.

[The expression "work done on it" is the common way of expressing the work done against the resistance, we say the work is done *on* the body moved.]

The spring is now a source whence effort can be derived, but not exactly a *natural* source, since we can only derive what we have originally put in. It is a simple example of an Accumulator, which is the name given in Mechanics to an apparatus in which energy is stored up, and from whence it may be drawn at will.

[The amount of energy we can obtain is not strictly equal to the work done, because there is an internal friction of the particles of metal, which wastes some of the energy or work, turning it into heat. Thus, whenever a spring is forcibly extended or compressed it is heated. This effect is, however, of small magnitude.]

Lifting of a Rope or Chain.—Fig. 64 represents a rope being drawn up and coiled on a barrel.

The resistance to motion is, at any instant, equal to the weight of rope then hanging—that already coiled having no effect.

The point of the chain which we will use to define the motion is the hanging end, because that keeps to a rectilinear path during the whole motion.

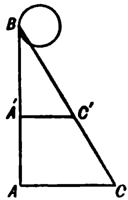


Fig. 64.

Let A represent the original position of the end, and AB the length $= l$ feet.

Let $w =$ weight in lbs. of 1 ft. length. Then

$$\text{Initial resistance} = wl \text{ lbs.}$$

Now, when A is raised to A′,

$$\text{Resistance} = \text{weight of BA}' = w \,.\, \text{BA}' \text{ lbs.}$$

Set off now, on a scale selected,

$$\text{AC} = wl \text{ lbs.}, \quad \text{A}'\text{C}' = w \,.\, \text{BA}' \text{ lbs.}$$

Then we have

$$\frac{\text{A}'\text{C}'}{\text{A}'\text{B}} = w = \frac{\text{AC}}{l} = \frac{\text{AC}}{\text{AB}}.$$

Thus, wherever we take A′, C′ lies on the line BC,

$$\therefore \text{ BC is the curve of resistance,}$$

and

$$\text{Work done during lift from A to A}' = \text{area ACC}'\text{A}',$$
$$= (\text{AC} + \text{A}'\text{C}')\,\frac{\text{AA}'}{2},$$
$$= (wl + w \,.\, \text{A}'\text{B})\,\frac{\text{AA}'}{2}.$$

For the whole lift $\text{A}'\text{B} = 0$ and $\text{AA}' = \text{AB}$,

$$\therefore \text{ Work done} = \frac{wl^2}{2}.$$

This result could also have been obtained directly by use of the method of page 62. Thus

$$\text{Total weight lifted} = wl.$$

Lift of C.G. is from the centre of AB to O, *i.e.* $l/2$ ft.

$$\therefore \text{ Work done} = wl \times \frac{l}{2} = \frac{wl^2}{2}.$$

We can also treat the question in still another way. For the chain barrel and its bearing form a turning pair.

Let, then, r = radius of barrel, and neglect the thickness of the rope.

Then when the end is at A′

$$\text{Resisting moment} = w \,.\, A'B \times r.$$

Therefore the moment varies as A′B, just as the resistance did; and the curve of moment is a straight line.

Let us then keep the same graphic figure and see what are the scales of it for this case.

AB now represents the total angle turned through. But to coil l ft. on the barrel we must turn it through an angle $\frac{l}{r}$,

$$\therefore \text{ AB represents } \frac{l}{r} \text{ of angle,}$$

whereas it formerly represented l feet. But AC now represents $wl \times r$ lbs.-ft., whereas it only before represented wl lbs. So that

$$\text{Work done} = \frac{AB \times AC}{2},$$

$$= \frac{\frac{l}{r} \times wlr}{2} = \frac{wl^2}{2} \text{ ft.-lbs.}$$

as before.

Elastic Fluid.—A vessel contains a certain quan-

tity of elastic fluid at a pressure P_2 lbs. per sq. ft., its volume being V_2 c. ft. Find the work done in compressing it to a volume V_1 and corresponding pressure P_1; or *vice versâ*, find the energy exerted by the gas in expanding from P_1 and V_1 to P_2 and V_2.

The work done depends on the curve of resistance, and this, as explained in chap. ii., page 49, we will take to be a hyperbola. For balanced forces this will also be the curve of effort.

In Fig. 65, 1 and 2 then represent the initial and final states of the fluid, taking expansion as the process to be considered (compare Fig. 41, page 48).

We will suppose the fluid confined in a cylinder under a piston—this being always the case in practice.

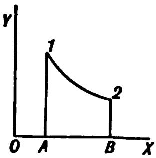


Fig. 65.

[This simplifies the work, but is not necessary to the truth of the result obtained.]

Now 1A represents P_1 lbs. per sq. ft. on a scale of say x ins. to 1 lb. per sq. ft. The effort or resistance, however, is not P_1 lbs. but $P_1 \times A$ lbs. where

$$A = \text{area of piston in sq. ins.}$$

But if the curve of P is a hyperbola, so also will the curve of $P \times A$ be, because A is a constant. But instead of x ins. representing 1 lb. per sq. ft. they will now represent 1 lb. per sq. ft. $\times$ A sq. ft. or A lbs.

Therefore 12 is the curve of resistance on the above scale.

Again, distances from O represent in the original state of the figure, *i.e.* Fig. 41, page 48, volumes in c. ft., *i.e.* OA represents V_1 c. ft. on a scale of say y ins. to 1 c. ft., so that OA is $V_1 y$ ins. long.

But OA will equally represent V_1/A ft., *i.e.* the distance of the piston from the end of the cylinder, if we take for our scale

$$V_1 y \text{ ins. to } V_1/A \text{ ft.},$$

or

$$y \text{ ins. to } 1/A \text{ ft.},$$

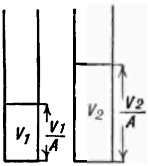


Fig. 66.

and then OB, on the same scale, represents V_2/A ft., for $OB = V_2 y$ ins. But V_2/A ft. is the distance of the piston from the end of the cylinder, when the volume is V_2 (Fig. 66).

It follows, then, that AB represents the path of the piston,

$$\therefore \begin{array}{l}\text{Energy exerted during expansion}\\ \text{or work done during compression}\end{array} = 1AB2,$$

$$= OA \times A1 \log_e \frac{OB}{OA} \quad \text{(p. 9)},$$

$$= \frac{V_1}{A} \times P_1 A \times \log_e r \text{ ft.-lbs.}$$

(r being the ratio of expansion or compression),

$$= P_1 V_1 \log_e r,$$

or equally

$$= P_2 V_2 \log_e r,$$

or generally

$$= PV \log_e r,$$

P and V referring to any point whatever during the expansion.

Indicator Diagrams.—Steam expanding in a cylinder does not follow the simple law just considered, nor can we determine the law of Effort, so that we are obliged to actually measure the effort at every point of the stroke. We have seen (page 44) how the pressure of an elastic fluid can be measured by allowing it to press on a small piston, thus compressing a spring.

Let now (Fig. 67) the end of the cylinder be in communication with such a small piston A, working in a cylinder *b*, and bearing against a spring confined between it and the cover of *b*, this cover being however perforated, so that the atmosphere has free access to the upper part.

To the piston a small rod is attached, passing through the cover, and having on its end a pencil.

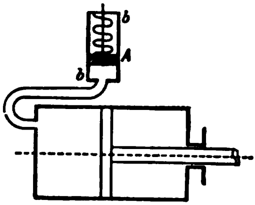


Fig. 67.

If now we cause to pass along, in contact with the pencil point, a piece of paper, *the motion of the paper being an exact copy, on a reduced scale, of the motion of the piston;* then the pencil point will trace on this paper a curve which gives the pressure of the steam at all points of the stroke.

For example, AB (Fig. 68) represents the piston stroke, *ab* the movement of the paper, EFG the curve traced by the pencil.

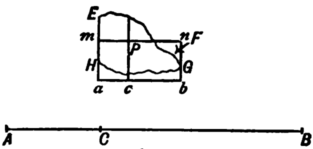


Fig. 68.

Then to find the pressure when the piston is at C, take *c* in *ab* as C is in AB, and measure the ordinate P from *ab*, the line of no pressure, and the same for any other point of the stroke.

ab then represents the movement of the piston, and EF is the curve of effort, *i.e.* of P × A, equally as of P, as in the preceding investigation. Whence the area of the curve gives the energy exerted.

We will now examine further into the question.

First, the paper is not moved straight along parallel to the piston, but is coiled on a barrel, which barrel is driven by a cord connected with the piston, so that the periphery of the barrel travels at a velocity bearing a constant ratio to that of the piston. Then at every instant the piece of paper *which is being marked* is travelling past the pencil with a motion bearing a constant ratio to that of the piston at the same instant. Evidently it does not matter where the remainder of the paper is.

Now, how do we determine the line of no pressure or the pressure axis? This is done by closing the connection between the small cylinder and the steam cylinder, and admitting the atmosphere below as well as above the piston. The piston then has atmospheric pressure below it, the amount of which we determine by the barometer. The paper barrel or drum is now rotated, and the pencil traces a straight line *mn*. Now *mn* represents say 14.7 lbs. per sq. in. This pressure corresponds to a given amount of extension or compression of the spring, this being known for each lb. pressure, being from $\frac{1}{12}''$, $\frac{1}{16}''$, up to $\frac{1}{120}''$, or less still, according to the magnitude of the pressures to be measured. We calculate then the amount the spring would be extended by the 14.7 lbs. pressure; and then setting off *ab* that amount below *mn*, *ab* is the line which would be drawn if there were no pressure at all in the cylinder, *i.e.* *ab* is the zero line. *mn* is called the atmospheric line.

The steam exerts energy during the forward stroke, but the piston must then return and drive out the steam behind it into the condensers, or atmosphere, if there be no condenser; and there is always a steam pressure, called the back pressure, resisting this return. Thus the pencil will, during a whole stroke, *i.e.* forward and back, trace a closed curve as EFGH, the paper still moving with the steam piston. During the forward stroke then the steam exerts energy represented by EF*ba*, while during the back stroke it absorbs a part G*ba*H. Thus on the whole the effective energy exerted is represented by the area of the figure EFGH.

The figure EFGH is called an Indicator Diagram, the instrument used being called an Indicator. The actual indicator differs from the simple form we have here described, the pencil not being directly attached to the rod, but to a multiplying gear, so as to increase the height of the diagram for a given travel of the indicator piston. Also the steam is admitted alternately on each

side of the main piston, and a connection is made from the indicator to each end of the cylinder, a three-way cock being fitted, so that each end may be in turn connected to the indicator. A diagram is drawn from each end on the same sheet of paper, and we thus obtain a card as shown in Fig. 69.

We do not directly determine the area of the diagram, and then, knowing the scales of length and of height, directly determine the energy; but we proceed by using the diagram to give us the Mean Effective Pressure. Strictly this would entail finding the area, but practically we proceed thus: Divide *mn* the length (Fig. 68) into ten equal parts, leaving half a part at each end (Fig. 69). (This is done because the shape at the ends is liable to distortion.) We thus obtain the ordinates.

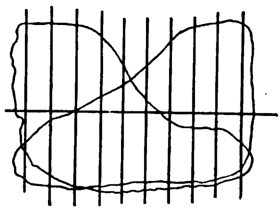


Fig. 69.

Measure these ordinates on the scale proper for the particular indicator spring, add them up and divide by 10.

The result is called the Mean Effective Pressure, because it allows for the loss of energy during the back stroke. Call it p_m. This is a pressure per sq. inch.

$$\therefore \text{ Effort} = p_m \times \text{area of piston in sq. ins.,}$$
$$= p_m \times A \text{ say,}$$

$$\therefore \begin{array}{l}\text{Energy exerted during whole stroke}\\ \text{on the one side of the piston}\end{array} = \begin{array}{l}p_m \times A \times s \text{ ft.-lbs.}\\ (s = \text{stroke in feet}).\end{array}$$

Then for the other side let p'_m be the mean effective pressure, and we get on that side

$$\text{Energy exerted during whole stroke} = p'_m \times A \times s \text{ ft.-lbs.}$$

$$\therefore \begin{array}{l}\text{Total energy exerted by steam}\\ \text{during a double stroke}\end{array} = (p_m + p'_m)As \text{ ft.-lbs.}$$

In practice p_m and p'_m are added together, and their

mean $\frac{p_m + p'_m}{2}$ is called the mean effective pressure of the card as a whole.

This mean effective pressure is to be supposed exerted through the whole double stroke. Whence

$$\text{Total energy in double stroke} = \frac{p_m + p'_m}{2} \text{A} \times 2s \text{ ft.-lbs.},$$
$$= (p_m + p'_m)\text{A}s \text{ ft.-lbs.},$$

as before.

Notice that, if P be per sq. ft., A must be in sq. ft.
if p be per sq. in., A „ „ sq. ins.

Comparing now with chap. iii., page 74, we have

$$\text{Horse power} = \frac{\frac{p_m + p'_m}{2} \times \text{A} \times 2s \times \text{N}}{33{,}000}.$$

[It is often now the practice to measure the area directly by means of an instrument called a Planimeter, but it is doubtful whether this gives any increase in accuracy, because there is often a difficulty in determining exactly the length of the diagram, which is of course necessary so as by division by it to obtain the mean breadth or mean effective pressure.]

EXAMPLES.

1. In question 3, page 74, if the throw of the pedals be 5 ins., and the driving wheel 54 ins. diameter. Find the mean pressure the rider exerts, assuming that he presses vertically during the downward half revolution of each crank. *Ans.* 15.27 lbs.

2. Assuming the resistance to a cutting tool to vary as the depth of cut. Compare the work done in planing a flat 2 ins. wide on an iron cylinder 4 ins. diameter, 12 ins. long, with that done in planing it down to the middle, and also with the work done in reducing its diameter to $3\frac{1}{2}$ ins. *Ans.* 1.82 : 32 : 15.

3. A spring-loaded safety-valve is $3\frac{1}{2}$ ins. diameter, loaded to 135 lbs. per sq. inch. The original length of the spring was 24 ins., and it is compressed to 20 ins. when fully loaded. Find the work done in lifting the valve through 1 inch.

Ans. 122 ft.-lbs.

4. A cylinder $6\frac{1}{2}$ ft. long, 2 sq. ft. sectional area, contains 1 lb. of air at atmospheric pressure. Find the work done in compressing it to four atmospheres according to the hyperbolic law.

Ans. 44,240 ft.-lbs.

5. In question 10, page 75, draw a curve showing the resistance at any point of the lift of the anchor—1st, neglecting buoyancy; 2d, taking account of the buoyancy of the water. S.G. of the iron 7.5. Calculate in each case the work done during each half of the lift.

Ans. 216,194$\frac{8}{3}$; 192.45; 180.12 ft.-tons.

6. If the engine of question 4, page 74, drive a pair of drums so that the rope unwinds from one while winding on the other. Draw a curve of resistance for one lift, assuming equal lengths of rope to wind and unwind in the same time. Weight of rope 8 lbs. per yard. Find the work done in each third of the lift.

Ans. 674,420, 166 ft.-tons.

7. The drums in the preceding are conical, varying from 20 ft. diameter to 30 ft., so that the winding and unwinding are not actually equal. Draw a curve of resisting moment due to the rope and coals lifted, and also a curve of the moment exerted by the descending rope; and by combining these obtain a curve showing the moment exerted at any point by the driving engine. Hence calculate the work done during each quarter lift. For simplicity take only the *complete* turns made.

Ans. There are 15 complete turns, so take 15 equal distances on the base line, on a scale of angle each representing 2π. The ordinates are then in order—

(21 cwt. $+\frac{8}{3}\times 375\pi$ lbs.) 20 ft.,
(21 cwt. $+\frac{8}{3}\times 355\pi$ lbs.) 20 ft. for 1st turn.

(21 cwt. $+\frac{8}{3}\times 355\pi$ lbs.) $20\frac{5}{7}$ ft.,
(21 cwt. $+\frac{8}{3}\times 334\frac{2}{7}\pi$ lbs.) $20\frac{5}{7}$ ft. for 2d turn.
etc. etc.

thus giving a stepped curve, assuming the rope wound in parallel circles of diameter 20 ft., $20\frac{2}{3}$ ft. etc., the moment changing suddenly at the commencement of each turn. Actually, the rope being wound continuously, the curve would be continuous, but the investigation is outside our present limits. The results are practically unaffected. A similar curve is obtained for the other rope, omitting the 21 cwt.

Results—

1st quarter, 618; 2d, 412;
3d, 206; 4th, 0 ft.-tons.

8. A steam engine is supplied with steam of 45 lbs. absolute, and discharges into the atmosphere, the back pressure being 17 lbs. absolute, and steam admitted during the whole stroke. Estimate the gain of work per cent: 1st, by fitting a condenser so that the back pressure is reduced to 3 lbs.; 2d, by further cutting off the steam at half stroke, so as to use only half the quantity per stroke, assuming expansion according to the hyperbolic law. *Ans.* 50; 70.

9. The ordinates of an indicator diagram are—

Front end 1.5, 1.48, 1.45, 1.1, .9, .75, .64, .6, .5, .3 ins.
Back ,, 1.6, 1.5, 1.3, 1.1, .85, .7, .66, .6, .45, .2 ,,

The indicator spring is compressed 1 inch by 16 lbs. Diameter of cylinder 45 ins., stroke 3 ft. 9 ins., revolution 65. Calculate the I.H.P. If the piston rod be 5 ins. diameter, and there be no tail rod, find what correction should be made in the preceding result. *Ans.* 341.7; 2.13.

10. The curve of stability of a vessel is a segment of a circle of radius twice the ordinate of maximum statical stability, which is 2500 tons-feet; estimate the total dynamical stability of the vessel, the angle of vanishing stability being 85°.

Ans. When a ship is forcibly heeled over, the water offers a resisting moment, commencing at zero, increasing to a maximum, and then decreasing again to zero, as the ship is heeled through an increasing angle; any heel will capsize her altogether. The curve of stability is a curve whose base shows angles of heel and ordinates corresponding resisting moments, so it is the curve of resistance to rotation; the resisting moment at any angle is called the Statical Stability, and the work done in heeling the ship to that angle the Dynamical Stability; the angle at which the statical stability vanishes is called the angle of vanishing stability. Hence the required dynamical stability is the total area of the curve of stability, viz. 2630 ft.-tons.

11. Calculate the H.P. of a turbine working for 10 hours a day, supplied from a reservoir at an elevation of 50 feet, containing 100,000 cubic feet, which is emptied at the end of 10 hours' work. The reservoir is continuously supplied by a stream which is capable of just filling it during the period of 14 hours rest. Assume .3 of the energy wasted. *Ans.* 20.5.

CHAPTER V

SIMPLE MACHINES

In order to do work we must have some source of energy, and in some instances the source can supply energy of exactly the nature required. For example, to draw a waggon we require say, a pull of 100 lbs. on the traces, this being the resistance offered by the road; then a horse is capable of exerting such a pull, and will move the waggon. In such a case

$$\text{Energy exerted} = \text{work done},$$

[In the actual case above it is all work wasted.]

and also

$$\text{Effort} = \text{resistance}.$$

But often the energy which the source can exert is of a different character to the work to be done. For example, to lift 1 ton through 1 foot,

$$\text{Work done} = 1 \text{ ft.-ton}.$$

Now a man is capable of exerting 1 ft.-ton of energy if he have time, but he cannot exert it in the form of an effort of 1 ton, but of say only 1 cwt. And then to do the above work an effort of 1 cwt. must be exerted through 20 ft. For the man therefore to lift 1 ton he must apply his effort to some part of an intermediate agent—say to the end of the rope of a set of pulleys—which agent is capable of changing the form of the energy to

that of the work, then the ton weight must be attached to some other point of the agent—in the case above to the other end of the rope—and the agent must be such as to magnify the effort 20 times, and also to reduce the velocity 20 times. Then will

1 cwt. exerted through 20 ft. lift 1 ton through 1 ft.

We say then that the man has to use a machine, and in most cases of doing work we have to use a machine, hence the necessity of the task which we now commence, *i.e.* an examination of the working of various machines.

The steam engine, for example, is a machine, transmitting the energy of the steam to the point at which we require work to be done, *e.g.* in a colliery winding engine, to the rope which lifts the cage or coals up the shaft, and in the transmission altering the character or nature from

Steam pressure × distance moved by piston

to

Weight of coals × distance through which they are lifted.

This of course implies that there is no waste of energy in transmission, which is never the case in practice; actually some will be wasted and the remainder transmitted.

The effect which the man seeks in using a set of pulleys is the magnifying of his effort, an effect which is expressed by saying the machine gives or has a Mechanical Advantage; the magnitude of the mechanical advantage being measured by the number of times the effort is increased, *i.e.* by the ratio of resistance to effort.

In old treatises on machinery these latter are called generally weight and power, but we have already given a definite meaning to power, and the resistance is not in all cases a weight, so we will keep to the terms resistance, effort.

Now although the mechanical advantage is what is sought, yet there is another effect which invariably

accompanies this, *i.e.* the alteration of velocity. We may not desire this but we cannot avoid it, and thus the consideration of the changes of velocity, or generally of the motion of the machine, becomes a part of our subject. In some cases, *e.g.* a watch, motion only is the requirement.

We shall commence now to examine certain machines, both as regards motion and as regards the forces to which the parts are subjected, due to the performance of work. We begin as usual with the simplest; and shall also in each case omit in the first place the influence of friction and take the case of Balanced Forces.

The Inclined Plane.—If we have a weight, say a waggon of weight W at A, which we wish to move to C, we might effect this by moving it horizontally to B and then lifting it direct to C.

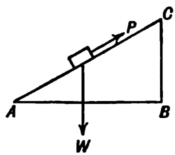


Fig. 70.

To do this, however, we require an effort to be exerted through CB equal to the weight W, and such an effort may not be available.

We then proceed by making an inclined plane from A to C, and we shall now see that an effort much less than W will be sufficient to effect the movement.

Let now the horse pull, exerting an effort in the direction AC. Let

$$P = \text{effort of horse.}$$

Because the forces are balanced, or because we suppose the only effect produced to be the movement of the waggon (compare page 60).

$$\text{Energy exerted} = \text{work done.}$$

But

$$\text{Energy exerted} = P \times AC,$$
$$\text{work done} = W \times CB \quad \text{(page 77)},$$
$$\therefore P.AC = W.CB,$$
$$\therefore P = W \cdot \frac{CB}{AC} = W \sin CAB.$$

The mechanical advantage then is

$$\frac{W}{P}=\text{cosec CAB or }\frac{AC}{CB}.$$

This we also call the **Force Ratio.**

Now to examine the motion. Let

V = speed of waggon along the plane.

[V may be uniform all along AC, or simply the velocity at D.]

Then V is the velocity of the effort.

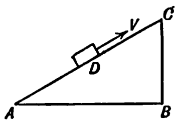


Fig. 71.

Now V is compounded of a horizontal velocity V cos CAB and a vertical velocity V sin CAB.

The velocity of motion then *against the resistance*, or the velocity at which the waggon is rising, is V sin CAB.

$$\therefore \frac{\text{Velocity against resistance}}{\text{Velocity of effort}}=\frac{\text{V sin CAB}}{\text{V}},$$
$$=\text{sin CAB.}$$

This is called the **Velocity Ratio**, and we get the relation,

$$\text{Velocity ratio}=\frac{1}{\text{Force ratio}}.$$

An equation which is equally true for all cases of balanced forces, and is in fact only another mode of expressing the Principle of Work. For—

$$\text{Energy exerted}=\text{work done}$$
$$\therefore \text{P}\times\text{movement of P}=\text{R}\times\text{movement of R},$$

whence it directly follows.

The work done will be the same, whether we start from rest and end up at rest, or start with a velocity which is kept uniform during the whole motion. The only condition necessary is that we leave off at C with the same velocity with which we commenced at A. The effort, however, would, for all cases except uniform

motion, vary, and instead of P we should have P_{mean} in the equations.

[The student must keep the preceding statement carefully in mind, as it applies to all cases of the present chapter, and we shall therefore not specifically state it for each case.]

The results obtained have been derived by use of the Principle of Work. We will now verify them by the use of other principles which will be found explained in all treatises on Statics.

The waggon W at D being in uniform motion, *the actions on it of bodies which tend to increase its motion must be exactly balanced by the actions of those bodies which tend to retard the motion.*

The statement here made differs somewhat in wording from that commonly made, which is that *the forces must be in equilibrium.* Now it is absolutely essential that the student recognise fully from the beginning that forces can only result from the actions of other bodies on that of which we consider the motion, and we have therefore stated the principle as above.

We proceed then to question ourselves as follows :—

(1) What is the body acted on, and whose motion is to be considered?

Answer—The waggon.

(2) What other bodies act on it?

The answer to this in the first instance is always the same, viz. all bodies which touch it. So we put the question in the form—What other bodies touch it?

Answer—The traces which pull it, and the plane.

But we must always include in addition a body which acts on it without touching it, viz. the Earth.

[We do not know *how* the Earth acts, but we do know it *does* by gravitation give rise to effort and resistance. Other bodies, as magnets, also can exert actions without touching, but such actions as these are outside our present scope.]

The full answer to (2) is then—

The traces, plane, and whole earth. The actions of these bodies are, in order,

P, R, and W.

Of these actions or forces R has no effect on the motion, being at right angles to it (compare page 76),

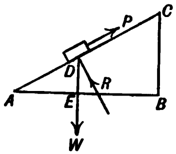


Fig. 72.

$\therefore$ P's effect = W's effect.

But P's effect is due to the whole of P, while W's is only due to the part of W acting in the line of motion, *i.e.* W cos EDA,

$$\therefore \text{P} = \text{W} \cos \text{EDA} = \text{W} \sin \text{CAB},$$

which verifies the result before obtained.

Here, in all probability, the student will think that a great deal of time has been spent in arriving at a result which could have been obtained in one, or at the most two lines by "resolving along AC."

But it is just by failing to put the questions we have considered that very many errors are made in the treatment of this subject. No doubt in the present simple case a correct result would be arrived at; but, in the most complicated questions possible, the student will, by putting these questions in order, necessarily arrive at a correct result, allowing of course for oversights or errors of calculation.

It is instructive to examine two different ways in which we might have answered question (1).

1st. We might answer—The waggon and traces. This is equally a body whose motion we can consider.

Then the answer to (2) becomes—The *collar*, plane, and earth; or, to (1), The waggon and all the harness; then to (2), The *horse*, plane, and earth.

It is perfectly immaterial which of the three methods of answering the questions we adopt, but we must take

one of the sets given complete. It would for instance not be correct to say in answer to (1) The waggon; and then to (2) The horse, because the horse does not touch the waggon, although he is of course in each case the natural source of the effort.

The student must clearly understand that it is only by thorough examination of the *details* of a question that we can arrive at a full understanding of it. And although such examination may in the present simple example appear even trivial, there are many cases in which neglect of such detailed examination leads to the most serious errors. We shall not be able from want of space to give this detailed process in every case, and shall use the ordinary expression "resolving the forces." But in every case the process will have been mentally gone through, which is the chief point; although, at any rate during the first study of the subject, the student will find it advantageous to actually write the process in full for each question.

There is one quantity which we have not found the value of because we could obtain our chief result without it, viz. the value of R. This is sometimes required, and to obtain "resolve in the direction of R" (see preceding). Then

$$R = W \cos CAB.$$

Case II.—Next let the effort be horizontal instead

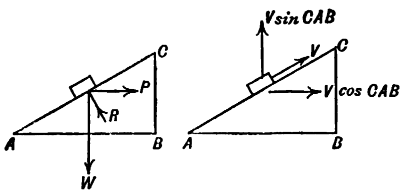


Fig. 73.

of along the plane. This is not a practical case in an

ordinary inclined plane, but we shall use it for a particular case in the next chapter.

We have then P, W, and R—the effort, resistance, and reaction of the plane. Then

$$\text{Energy exerted} = \text{Work done},$$
$$\therefore \ P \times AB = W \times BC \quad \text{(page 77)},$$
$$\therefore \ \text{Force ratio} = \frac{W}{P} = \frac{AB}{BC} = \cot CAB,$$

and the inverse is the velocity ratio, for—

$$\text{Velocity of P in its own line of action} = V \cos CAB,$$
$$\text{,, \quad W \quad ,, \qquad ,,} \qquad = V \sin CAB,$$
$$\therefore \ \text{Velocity ratio} = \frac{V \sin CAB}{V \cos CAB} = \tan CAB,$$
$$= \frac{1}{\text{force ratio}}.$$

To obtain R we resolve vertically, which gives

$$R \cos CAB = W,$$
$$\therefore \ R = W \sec CAB,$$

so that R is in this case greater than in Case I., the reason being that P helps to pull the slider or carriage against the plane.

The Wheel and Axle.—In this machine we utilise the turning pair to magnify an effort, which effect we produced in the inclined plane by the use of a sliding pair.

The motion has been already fully investigated in the preceding chapters, so we need little further description.

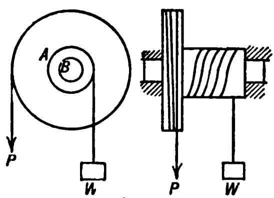


Fig. 74.

In Fig. 74 A is the wheel, B the axle, these forming one solid piece, the fixed frame, containing the bearings, forming the other.

The effort P acts on the end of a rope, coiled round, and fastened to a point in

the wheel. The resistance W acts on a rope similarly connected to the axle.

If r_1 r_2 be the radii of the wheel and axle respectively, the condition for balanced forces is (page 65),

$$Pr_1 = Wr_2,$$
$$\therefore \text{ Force ratio} = \frac{W}{P} = \frac{r_1}{r_2}.$$

For the velocity ratio take one revolution, then

$$\text{Lift of } W = \text{length of rope coiled on axle,}$$
$$= 2\pi r_2.$$
$$\text{Fall of } P = \text{length of rope uncoiled from wheel,}$$
$$= 2\pi r_1.$$
$$\therefore \text{ Velocity ratio} = \frac{2\pi r_2}{2\pi r_1} = \frac{r_2}{r_1} = \frac{1}{\text{force ratio}}.$$

If the axis be horizontal we have the common windlass (Fig. 75), if vertical the capstan (Fig. 76). In the

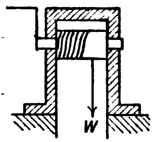


Fig. 75.

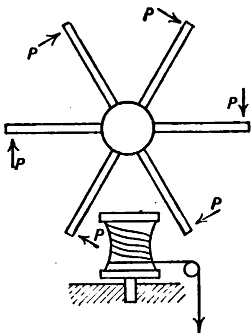


Fig. 76.

first the effort is applied to a handle, which takes the place of the wheel, but comparing with chap. iii., page 64, we see that the shape of the piece does not affect the motion.

In the capstan the effort or efforts is applied to bars radiating from the head, as shown in the plan ; the rope

lifting the weight passes away horizontally, and its direction is changed to vertical or any other required by passing it over guiding pulleys.

These two cases supply fresh examples of the application of a single force and of a couple. The windlass requires stout bearings, but the capstan being moved by forces applied in pairs at the ends of opposite bars, *i.e.* by couples, can do without any bearing at all at the top (compare page 66).

The Screw.—To use this, the third simple pair, as a machine, we apply a moment to effect the turning motion, either to the screw or nut, the other piece being prevented from turning; and we apply the resistance to resist the sliding motion of one piece, the other being prevented from sliding, *i.e.* the effort causes the *relative* turning and the resistance resists the *relative* sliding.

The double nature of screw motion makes this case a little more complicated than the preceding ones. In the inclined plane we generally regard the plane as a fixed body, since it is connected to the earth in nearly all practical cases. Also in the wheel and axle the bearings are considered as fixed; in the case of the windlass generally to the earth, and in the case of the capstan to the earth or to a ship; in either case they are fixed relatively to the observer.

But in the screw motion not only may either be actually fixed relatively to the observer, but one may be fixed against turning yet free to slide, while the other is fixed against sliding while free to turn, or *vice versâ;* the turning and sliding here mentioned being relative to the observer.

None of these considerations, however, will affect the *relative* motion, and hence we can at once apply the Principle of Work to give us the relation between the effort and resistance, and then examine some practical cases showing the different ways in which the pair is used.

Let then

$$M = \text{turning moment,}$$

and for definiteness we will take the nut to be fixed relatively to ourselves, the observers.

$$R = \text{resistance to sliding.}$$

Then, taking a period of one revolution,

$$\text{Energy exerted} = M \times 2\pi,$$
$$\text{Work done} = R \times p,$$

where p = pitch of screw, and therefore is the amount of sliding against R (page 34),

$$\therefore\ M \times 2\pi = R \times p.$$

This then is the general equation, and is quite independent of which piece is fixed, depending, as it does, entirely on relative motion.

We can now if we please suppose M to be applied by an effort P, acting at a radius r. Then

$$M = Pr,$$
$$\therefore\ P \times 2\pi r = R \times p.$$

We may then say

$$\text{Force ratio} = \frac{R}{P} = \frac{2\pi r}{p},$$

and

$$\text{Velocity ratio} = \frac{p}{2\pi r} \quad \text{(page 34).}$$
$$= \frac{1}{\text{force ratio}}.$$

The value of the force ratio shows that these machines are suitable where the effort requires to be much magnified, for example, in a screw jack (page 36).

Let a man apply an effort of 40 lbs. to the end of a handle 2 ft. long, then to find the resistance he can overcome we have

$$R = 40 \times \frac{2\pi \times 2}{p} = \frac{160\pi}{p} \quad (p \text{ in ft. since } r \text{ is}).$$

Now p will be say $\frac{1}{2}$ in., *i.e.* $\frac{1}{24}$ ft.

$$\therefore\ R = 24 \times 160\pi \text{ lbs.,}$$
$$= 12{,}000 \text{ lbs. about.}$$

[Of course friction will greatly modify this in an actual case, see next chapter.]

We considered the screw jack in chap. ii., page 36; looking back we see it is a case where the nut is fixed—since, though not actually fastened to the ground, its weight keeps it practically so—while the screw both turns and advances, R is W the weight lifted, and M is applied to the screw.

Screw Press.—The screw press is another example

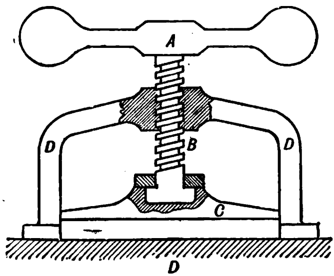


Fig. 77.

of the same type. The moment is applied to the handle A, which is fixed to B the screw, while the article to be pressed lies between the piece C which slides in the frame D, and the bottom of D.

R is then the pressure applied to the article pressed.

In both the screw jack and the press there is a part which will repay a little examination. It is practically identical in each, so we will take the case we have just considered, viz. the press.

The source of resistance is some body between C and D, and this body offers its resistance to C, which it touches, not to B, the screw. We have then strictly no

right yet to say that R is the pressure applied to, or the resistance offered by, the body pressed.

But, considering the motion of the sliding pair CD, we have

Effort = resistance.

But the effort on C is exerted by the end of B, and is therefore exactly equal and opposite to the resistance which C offers to the end of B,

$\therefore$ If R = resistance to advance of B,
R = effort acting on C,

and therefore also

R = resistance offered by the body pressed to the sliding of C.

But the resistance which the body offers to C *is equal and opposite to the pressure which* C *exerts on the body;* and hence then the value of this pressure is R, as we have stated it to be.

The effect then of C is nothing so far as ratio of pressure to moment is concerned; C being fitted, not for any reason connected with energy or work, but simply to prevent the end of B rubbing, by its turning motion, the body to be pressed. Although, therefore, we have in reality a machine containing three pairs, viz. a screw pair BD, a turning pair BC, and a sliding pair CD—yet we may practically treat it as if it were a simple screw pair only.

We have just been using, for the first time, a very important principle; which, while practically self evident, is yet liable, if not thoroughly comprehended, to cause confusion. The principle we refer to is that ordinarily stated in the form

Action and Reaction are equal and opposite.

We have used the principle twice in proving that R, the resistance to the screw motion, is also the pressure on the body pressed. We will take the one case which we have italicised above for further consideration.

Stress—Force.—Now what we really know about C and the body pressed is that there is a mutual action between them where they touch. This is only one action, but it has two aspects, according, we may say, to our point of view; we can say C exerts a pressure on the body—or, the body exerts a pressure on C. Each of the preceding statements is true, and moreover neither can exist alone, so that they are not two, but one as we have said.

The action being one has one name, which applies to it as a whole, and we say a **Stress** exists between the surfaces of C and the body. But since there are two ways of looking at the action, we have also a name for each of its aspects, viz. **Force**. So we say C exerts a force on the body, and the body exerts a force on C.

The true and full statement of the action between C and the body then is that a stress of magnitude R lbs. exists between them, and since no way of looking at the action can alter its value, it follows that the force exerted by C on the body, and that exerted by the body on C, are each R lbs.

The two forces just mentioned are those which are sometimes called Action and Reaction. The reason for giving the name Action to the first more than the second being that C is generally the moving or acting body, while the pressed body lies still. But so long as we deal with relative motion, and treat the two bodies as on an equal footing as regards motion, we need not distinguish between the two.

[There is no difficulty in making C the passive body if we like; we have only to fix it, in a vice say, and then we should screw the pressed body up against it. We should certainly in no way affect the mutual action between them by doing this.]

It is not only between bodies which actually touch that a stress can exist. For we shall find that every case in which one body exerts a force on another is a case of stress between the two. Take gravity for in-

stance. We say the earth exerts a pull on a falling weight, but what we really know is that they have a mutual tendency to approach each other, *i.e.* there is a mutual action, a stress, between them. The stress we can describe either as the earth pulling the weight, or the weight pulling the earth; and it is impossible to say that one of these is more or less correct than the other.

Similarly for a magnet and a piece of iron, or any instance whatever of what we call forces.

We can now define strictly a term which we have hitherto used without defining, viz. Force. We have not as yet attempted this, because the meaning is quite sufficiently understood for all the work we have done, and the introduction of too many definitions at the commencement of a subject is as likely to confuse as to clear our ideas.

We say then

Force *is one part of the mutual action or stress between two bodies, either in contact or not.*

The reasons for this statement have just been given.

[It is not necessary for us to consider the *cause* of force, or *how the earth can exert force on a body it does not touch.* There is no doubt that force is present, and it is quite time enough to inquire the cause when we have fully investigated all the effects which follow from its presence.]

The Screw Propeller.—This is a similar case to the one just considered. The screw is turned, the nut, viz. the water, being fixed. The water, however, as well as taking the place of the nut D, also supplies, by friction and otherwise, the resistance to the ship's motion. The ship and thrust block take the place of C, and the propeller with its shafting that of B.

The ship, being in uniform motion, is acted on by the resistance of the water R, and by an equal force applied to the thrust block by the shaft. This force is called the Thrust.

So the equation of energy becomes

Turning moment on shaft × 2π = thrust × pitch of propeller.

Other Examples.—As an example of a fixed screw and moving nut we may take an arrangement used by Whitworth in making compressed steel shafts.

A is the mould, BB are screws, and the cross-piece C is run down by turning the nuts DD at about 500 revolutions per minute. The projection on the cap fits into the top of the mould, and when the mould has been filled and C run down, the compression is produced by hydraulic pressure in the space inside the vessel E, forcing the mould up against C. The mould shown is for a piece of hollow shaft.

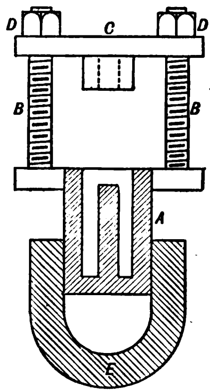


Fig. 78.

In the motion of the saddle of a screw-cutting lathe we have each piece partially fixed, for the leading screw can turn in its bearings but not move lengthways, while the saddle can slide along the bed of the lathe but not turn. The effort is applied to the screw, and the resistance to the point of the tool which is rigidly attached to, and therefore is for the time a part of, the saddle.

The Screw as an Inclined Plane.—Looking back to page 103 we have the equation

$$P \times 2\pi r = R \times p.$$

Now this equation is the same as that for an inclined plane of height p, base $2\pi r$, the effort acting horizontally (page 95).

Hence the velocity ratio also must be the same, which is easily seen to be the case, and in fact the cases are identical, except that the horizontal motion of the resist-

ing body, *i.e.* the slider in one case and nut in the other, is in the plane all in one direction, while in the screw it is in a circular path. Now this alteration can be effected by coiling the plane round in a cylinder of radius r (Fig. 79), and then its slant side takes the form of a thread.

It is usual to consider the screw as derived in this

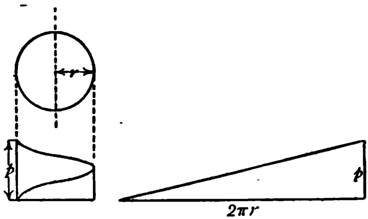


Fig. 79.

way, but we have preferred to obtain its properties by consideration of the actual mode of constructing it in practice. This mode, however, we must notice, presupposes the existence of another screw, viz. the leading screw of the lathe, while the present gives a screw independently of any previous screws.

We have here considered a screw as being virtually an inclined plane, in order to show the identity of the two ideas, but we shall not now use this new conception, since we do not gain by it. It will, however, be required in chap. vii.

EXAMPLES.

1. The draught of a waggon is 40 lbs. per ton. Assuming this to be constant irrespective of slope, compare the speed with which a horse could draw the waggon on a level with that with which he could pull it up a slope of 1 in 50; exerting energy at the same rate in each case. *Ans.* 2.12 : 1.

2. If the waggon above weigh $2\frac{1}{2}$ tons, find the greatest slope

up which two horses could pull it, supposing they can each exert a pull of 150 lbs. *Ans.* 1 in 28.

3. Why does a horse zigzag when pulling a load up a steep hill?

A loaded cart weighs 1 ton, constant resistance 45 lbs. The horse can only exert a pull of 112 lbs.; how many times must he cross the road in going up a hill 150 yards long, rising 1 in 25, width of road 35 ft.? *Ans.* 12.

4. A truck weighing $2\frac{1}{4}$ tons rests on an incline at 30° to the horizontal. It is fastened up by a rope 6 ft. long, fastened to a hook in the truck 3 ft. from the ground, and connected at the other end to, 1st, a fixed point 3 ft. from the ground; 2d, a point on the ground. Find in each case the tension of the rope, and the pressure on the incline.

Ans. Tensions, $1\frac{1}{8}$, 1.3 tons; pressures, 1.97, 2.6 tons.

5. In question 10, page 75, the capstan is 3 ft. diameter. How many men would be required to turn it, each exerting a push of 40 lbs., and the distance of the resultant push on each bar from the centre of the capstan being 8 ft.? *Ans.* 75.

6. The pitch of a screw propeller is 14 ft., and the twisting moment applied to it is 120 tons-inches. Find the thrust.

Ans. $4\frac{1}{2}$ tons.

7. In question 9, p. 38, what force applied to the handle will lift 1 ton? *Ans.* $8\frac{1}{4}$ lbs.

CHAPTER VI

PULLEYS, BELTS, AND WHEEL GEARS

THE simple machines already considered have consisted practically of one pair. We will now consider some cases of the connection of two pairs.

Pulley Blocks.—The pairs here connected are not real but virtual pairs. Taking the case of a small weight lifting a large one, each weight forms a virtual sliding pair with the earth, and the pairs are connected by the pulley blocks and ropes, so that motion of the one causes a certain motion of the other. If the end of the fall, *i.e.* the part to which the effort is applied, be pulled in some other way than by a weight, there are some means generally by which it is guided in a straight path, and then any piece of it may be considered as forming, with the earth, a sliding pair.

[By the above manner of consideration the wheel and axle and screw are also connections of pairs. There is, however, a further difficulty in pulleys, due to the rope connection, hence we place them in this chapter.]

Thus in Fig. 80, which is the simplest of all pulleys, the piece between P and the pulley may be taken as forming a sliding pair with the earth, being connected where it meets the pulley to the rope, and the effort applied by the hand say which is applying the effort P.

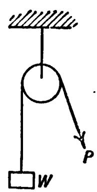


Fig. 80.

[It may seem strange to describe the piece of rope as being connected to the rope, because it is a part of the

latter, but it certainly is connected, and in fact by the closest of all possible connections.]

A set of pulleys, or of blocks, as they are usually called, consists of a rope or ropes passing round small wheels called sheaves, which rotate on pins. Now we must inquire why these sheaves are fitted, and if their motion relative to the pins forming a turning pair has any effect on the energy or work.

It is a very common use for one pulley or sheave to place it as in Fig. 81 (*a*), in order to change the direction of a rope which passes over it. Now this effect could equally be obtained, as in (*b*), by passing the rope over a rounded surface; but then there would be considerable friction as the rope slid over the surface. The sheave then is fitted to avoid this friction, and now there is no slipping between the rope and sheave, but all the relative motion takes place at the surface of the pin, and thus the friction is very much reduced. The reason for fitting the sheave then is to change the direction of the rope without undue friction; but, being fitted, has it any effect in modifying the *tension* of the rope?

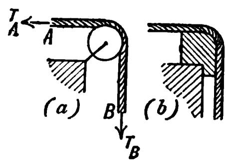


Fig. 81.

We are not considering friction at present, so we suppose the motion of the sheave on the pin to be frictionless, and in this case the answer to the question just asked is No. For let the motion be in the direction of the arrow, and consider the piece of rope AB as a body acted on by tensions T_A, T_B at A and B respectively, and by the pressures of the pulley. These latter are everywhere normal to the pulley, because, since the pulley turns uniformly, and the pin being frictionless can exert no moment on it, it follows that the rope can exert no moment on it, so that the pressure of the rope on the pulley must have no

moment, *i.e.* everywhere be normal, and hence so also are the pressures of the pulley on the rope (page 106).

These pressures then are everywhere at right angles to the motion of the piece of rope they act on, and can therefore have no effect on the motion. Hence then T_A and T_B are effort and resistance, and there are no other forces.

But velocity of A = velocity of B, so the velocity ratio is unity, and therefore so is the force ratio,

$$\therefore\ T_A = T_B.$$

We have then the principle that in the absence of friction the tension of a rope is unaltered by passing round a pulley, and the work we have done will not be affected by any motion of the pulley, so long as no moment be applied to it, *i.e.* we may apply any force we please through its centre without affecting our equations.

We see then at once that we cannot obtain any mechanical advantage by the use of a single fixed pulley, *i.e.* pulley with fixed centre, as Fig. 80, for we have by our principle

$$P = W.$$

But now in addition to a fixed pulley let us take a movable pulley.

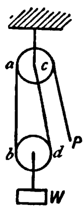


Fig. 82.

W is not now fastened to the rope to which P is applied, but to the framework of the movable pulley, and the rope, after passing round both, is led up and fastened to the frame of the fixed pulley.

Let P be drawn down say 2 ft., then W rises, shortening both *ab* and *cd*, and neglecting the little deviation from parallelism they shorten equally; so each shortens 1 foot, which is therefore the rise of W,

$$\therefore \text{ Velocity ratio} = \frac{2 \text{ ft.}}{1 \text{ ft.}} = \frac{2}{1}.$$

Therefore the force ratio $= \frac{1}{2}$, and

$$P = \frac{W}{2},$$

the mechanical advantage being 2.

We will now verify this.

W is supported by *ac* and *bd*. The tensions in these are equal, and each equal P (see previous principle),

$$\therefore W = 2P,$$

as above.

The actual construction of a block is shown here (Fig. 83), the hooks being for the attachment of ropes, so that this may be hung up to a fixed point, forming the fixed pulley, or W be hung to the hook, and it can

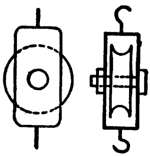


Fig. 83.

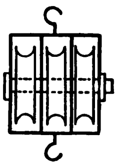


Fig. 84.

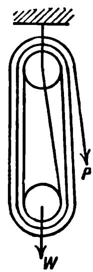


Fig. 85.

form the movable pulley. A pair of such blocks with the rope which goes round the sheaves is called a tackle, or system of pulleys.

If we desire to still further increase the mechanical advantage, we can do so by using more than one sheave, say for example three as here shown (Fig. 84).

We use a pair of such blocks, and call the whole a pair of three-sheaved blocks. The rope would pass in turn round an upper and under pulley, being finally fastened to the lower hook of the top or fixed block.

Fig. 85 shows a diagrammatic representation of the run of the rope.

We will now examine the general case when there are n plies of rope supporting the lower block.

[n may be odd or even; in a pair of three-sheaved blocks it is 6; but we may have 3 sheaves at top and 2 at bottom, then the final fastening would be to the top hook of the lower block, and there would be 5 plies.]

By our principle the tension all through the rope is P, therefore nP supports W, and

$$W = nP,$$

or

$$\text{Force ratio} = n.$$

Also if P move a distance x, all the n plies shorten equal amounts, so that each shortens x/n, the rope remaining of unaltered length.

x/n then is the rise of W, and

$$\text{Velocity ratio} = \frac{x/n}{x} = \frac{n}{1},$$

verifying our result.

All sorts of combinations of pulleys can be used for various purposes, but the same principle applies to all, and so we shall not examine their working; moreover, the one we have considered is of far more importance than all the rest together.

Belt Connection.—Next we will consider the connection of two turning pairs.

We have generally so far considered the magnifying of the effort as the effect sought after, but inseparably connected with this we have seen there is a modification of the velocity. For in all cases

$$\text{Force ratio} = \frac{1}{\text{velocity ratio}}.$$

In some cases it is the modification of velocity which is chiefly aimed at, the alteration of effort which necessarily follows being regarded as of subsidiary importance, or even in some cases, *e.g.* the mechanism of a watch, of no importance at all. In this case we

should not call the watch a machine but a mechanism, its sole object being the production of a certain motion.

We shall then treat the present case, in the first instance, from the view of modification of motion, and deduce, when necessary, the accompanying change of effort. And this method has the advantage that our main treatment, being purely geometrical, will be just as true when we have to take frictional forces into account as now when we omit them, which would not be the case if we based it on the modification of the effort.

[This will be seen in the examples considered in the succeeding chapter.]

We have now then this problem :—

Given two turning pairs, *i.e.* two shafts turning in bearings fixed to the earth or to some framework, it is required to connect them so as to have a given velocity ratio.

1st Case.—Where the distance apart of the centres is large.

A and B are the two shafts, the bearings are not shown.

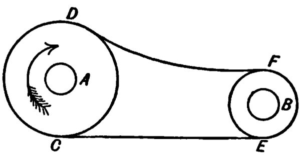


Fig. 86.

Fix now pulleys CD and EF on A and B respectively, and connect these by an endless belt or rope passing round them as shown. Then as A turns the belt turns with the pulley CD, and so causes the rotation of EF, *i.e.* of B.

Let now

$$A_A = \text{angular velocity of A},$$
$$A_B = \text{required angular velocity of B}.$$

The motion being turning, the velocity ratio will be one of angular velocities. Let

$$r_A = \text{radius of CD},$$
$$r_B = \text{radius of EF}.$$

The principle governing the connection is that the

total length of the belt is constant, also the lengths of the parts CE, EF, FD and DC, and that the belt does not slip on the pulleys.

Since the belt does not slip on CD,

$$\begin{aligned}\text{Speed of point C of belt} &= \text{speed of point C of pulley,}\\ &= \text{speed of any point on periphery,}\\ &= \mathrm{A_A} \cdot r_{\mathrm{A}} \quad \text{(page 31).}\end{aligned}$$

Similarly from EF

$$\text{Speed of point E of belt} = \mathrm{A_B} r_{\mathrm{B}}.$$

But CE is of constant length,

$$\therefore \text{ Speed of C} = \text{speed of E,}$$

or

$$\mathrm{A_A} r_{\mathrm{A}} = \mathrm{A_B} r_{\mathrm{B}},$$

whence

$$\frac{r_{\mathrm{A}}}{r_{\mathrm{B}}} = \frac{\mathrm{A_B}}{\mathrm{A_A}}.$$

That is, the angular velocities of the shafts vary inversely as the radii of the pulleys. We must then take r_{A} and r_{B} such as to satisfy the above relation.

The shaft A may represent the main shaft of a factory, and we see that by fitting to it pulleys, which drive by belts other pulleys on shafts fitted to different machines, we can from the one shaft obtain any number of different angular velocities in the other shafts.

Length of Belt.—There are two ways in which the

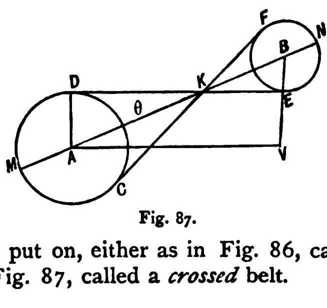


Fig. 87.

belt may be put on, either as in Fig. 86, called an *open* belt, or in Fig. 87, called a *crossed* belt.

In the first case both shafts rotate in the same direction, while in the second they rotate in opposite ones.

There is a peculiarity connected with the length of a crossed belt which is of practical importance.

In Fig. 87 A and B are the centres of the pulleys.

Produce AB both ways to M and N.

Let θ be the circular measure of DKA. Then

Half length of belt

$$\begin{aligned}
&= \text{MD}+\text{DK}+\text{KE}+\text{EN},\\
&= r_A \times \angle \text{MAD} + r_A \operatorname{cotan}\theta + r_B \operatorname{cotan}\theta + r_B \angle \text{EBN},\\
&= r_A\left(\frac{\pi}{2}+\theta\right) + (r_A+r_B)\operatorname{cotan}\theta + r_B\left(\frac{\pi}{2}+\theta\right),\\
&= (r_A+r_B)\left(\frac{\pi}{2}+\theta+\operatorname{cotan}\theta\right) \qquad (1).
\end{aligned}$$

Also, producing BE, and drawing AV parallel to DE, *i.e.* perpendicular to BE, we have

$$\theta = \angle \text{BAV}, \quad \therefore \sin\theta = \frac{\text{BV}}{\text{AB}},$$
$$= \frac{r_A+r_B}{\text{AB}}.$$

Therefore if $r_A + r_B$ and AB be constant, θ is constant; and therefore by (1) the length of the belt is constant.

Speed Pulleys.—Let now B be a shaft, which we require to drive at different velocities, from a shaft A which turns at a fixed velocity. We can effect this by fixing to A a set of pulleys of different diameters, and to B a corresponding set.

We thus obtain a pair of *speed pulleys*, as Fig. 88, and it appears from what we have just proved that so long as each pair of corresponding pulleys have

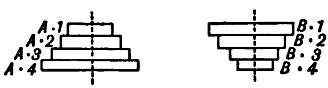


Fig. 88.

the sum of their radii constant, a crossed belt can be

shifted from any one pair to any other, and will fit them all equally well. This is not true for an open belt, and hence we almost invariably find a crossed belt used; the open belt would drive, but would be tight on some and loose on others of the set.

The pulleys fixed to A are all cast in one, and likewise those fixed to B.

Suppose now we wish to find the sizes of such a set to impart to B certain given velocities.

Let A_A = constant angular velocity of A, and $A_{B.1}$, $A_{B.2}$, $A_{B.3}$, $A_{B.4}$ be the required velocities of B.

We should then have four pulleys, as in Fig. 88, on each shaft, or we should say four steps on each pulley. Call them A.1, A.2, etc., B.1, B.2, etc., as in the figure, A.1 and B.1 corresponding, and so for the others.

Let $r_{A.1}$ . . ., $r_{B.1}$ . . . be the radii. Then

$$\frac{r_{A.1}}{r_{B.1}} = \frac{A_{B.1}}{A_A},$$
$$\frac{r_{A.2}}{r_{B.2}} = \frac{A_{B.2}}{A_A}.$$

And the same for 3 and 4, we do not put $A_{A.1}$ or $A_{A.2}$, because A as a whole has always the velocity A_A.

In addition to the above

$$r_{A.1} + r_{B.1} = r_{A.2} + r_{B.2} = r_{A.3} + r_{B.3} = r_{A.4} + r_{B.4}.$$

And now if we are given the size of any one step we can find all the others. We must have one radius given, because evidently, if we say doubled the sizes of a given set right through, we should not affect the velocity ratios; our equations above will only give us, by themselves, the ratios of all the radii.

Belt Tensions.—Returning to the original case (Fig. 86), let us consider the relations of effort and resistance. Let

M_A = moment of effort applied to A,
M_B = " resistance " B.

Then taking the whole system,

$$\text{Force ratio} = \frac{1}{\text{velocity ratio}},$$

$$\therefore \frac{M_B}{M_A} = \frac{A_A}{A_B} = \frac{r_B}{r_A}.$$

Next consider the motion of A only, taking, however, as a part of A the piece of belt which touches it.

[This we can do, because the piece moves as if it were for the time solid with A, and since it *does not* move relative to A, it is in just the same state as if it *could not.*]

The body we consider, then, is balanced under the action of M_A, the tension of the belt at C, and the tension at D. Call these T_1 and T_2, as in Fig. 89, then we have

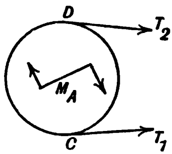


Fig. 89.

$$M_A + T_2 r_A \text{ (clockwise)} = T_1 r_A \text{ (anti-clockwise)},$$

$$\therefore (T_1 - T_2) r_A = M.$$

There is an essential difference then between this case and the pulley sheave (page 112), for T_1 and T_2 are necessarily unequal. This is because a moment is applied to A (compare pages 112, 113).

Now considering the balance of forces on the piece of belt DF (Fig. 86), we have

$$\text{Tension at F} = T_2,$$

and on the piece CE

$$\text{Tension at E} = T_1,$$

which accordingly gives us a turning moment applied by the belt to B of amount

$$(T_1 - T_2) r_B,$$

which is of course equal to M_B. For

$$M_B = M_A \cdot \frac{r_B}{r_A} = (T_1 - T_2) r_A \times \frac{r_B}{r_A} = (T_1 - T_2) r_B.$$

Since $T_1 > T_2$ the belt hangs as shown, the under side being tight and the upper slack.

In this case A is called the driver, because the effort is applied to it. If B were the driver, and the motion were, as in the present case, clockwise, then the upper side would be tight, as Fig. 90. This latter is not so good as the former for a practical reason, which we must now briefly notice.

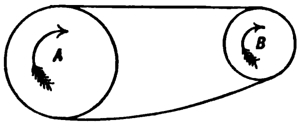


Fig. 90.

We have assumed the belt not to slip on the pulley; this is in practice not quite accurate, there will be a small amount of slip depending on the magnitudes of T_1, T_2 and the moments, also on the state of the surfaces. Now a belt, as in Fig. 86, embraces a larger portion of each pulley than that in Fig. 90, and this gives it a greater driving effect and reduces slipping. The effect of slip is that B moves somewhat slower than we have reckoned, the loss varying from practically *nil* in some cases to about 2 % in others.

The belt is pressed against the pulley by a resultant force $T_1 + T_2$, and this also gives the pressure of the shaft on its bearings.

The value of $T_1 + T_2$ must be large enough to prevent the belt slipping to any extent, and then $T_1 - T_2$ being found from the value of M_A, we can determine the two tensions T_1 and T_2. Since applying a moment to A or B cannot produce a resultant force on them (page 66), it follows that when the belt is still the total pull on each pulley must be $T_1 + T_2$. But it will then be equally divided between the two sides of the belt, since otherwise there would be a turning moment.

We should have then when at rest the tension of each side equal $(T_1 + T_2)/2$, and this therefore is the tension with which the belt must be originally put on.

Friction Wheels—2d Case.—When the shafts

connected are close together we can connect them by causing the surfaces of the two pulleys to rub together instead of putting a belt round them.

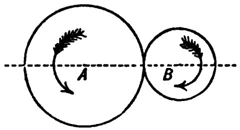


Fig. 91.

Then A will drive B by the contact of their surfaces, B rotating in the opposite direction to A.

For the velocity ratio we have, *if the surfaces do not slip*,

Velocity of periphery of A = velocity of periphery of B.

If then V be this common velocity

$$A_A = \frac{V}{r_A}, \quad A_B = \frac{V}{r_B}, \quad \text{and } \frac{A_B}{A_A} = \frac{r_A}{r_B},$$

as in the belt connection.

The contact between the surfaces of A and B is called pure rolling, since the surfaces roll without slipping. In actual practice there is always slipping, yet the contact is still called rolling, but not pure rolling. If the energy exerted on A and transmitted by the contact to B be large, then a very great pressure between the surfaces is necessary to prevent excessive slipping; this kind of contact is not therefore suitable for such cases. Since the pulleys, however, run very smoothly, they are sometimes used with rims shaped as here shown (Fig. 92).

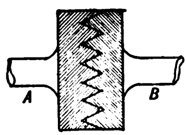


Fig. 92.

Less pressure is then required between the surfaces owing to the increased area of contact. The figure shows a section through the two rims.

There must of necessity be slipping at the different points in this case, even if on the whole there be none. For take one groove enlarged, as Fig. 93. Then if at *a* the two surfaces have exactly the same velocity V, it follows at once that at any other point they slip. For

take b, the surface of B is here moving at a speed $<$V, because b is nearer the centre than a; but the surface of A has a speed $>$V, because b is farther than a from the centre of A, hence the two surfaces have different speeds at b, or at all points except a. It is this which causes the rapid wear of such wheels.

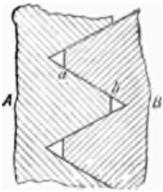


Fig. 93.

Tooth Wheels.—In any case then in which accuracy of velocity ratio is of importance we must devise some other mode of connection, and this we do by putting teeth on the circum-

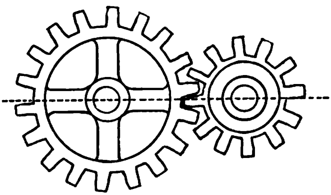


Fig. 94.

ferences of the wheels or pulleys. The teeth project partly beyond the circumference, CD showing the original circumference of a flat-faced wheel or pulley; we have then alternate projections and recesses on the circumference of A, which fit into corresponding recesses and projections on the circumference of B. The teeth on A and B are, almost invariably, of identical shape. The distance CE on the arc is called the pitch, and must be identical for A and B. Now by this means A will turn B as in pure rolling. For let A turn till n teeth of it have passed the point T (Fig. 96) where the original plane pulleys touched.

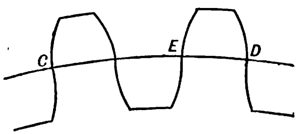


Fig. 95.

T is called the pitch point, and the circles showing the original pulleys the pitch circle. Then if p be the pitch, an arc np of B has passed T.

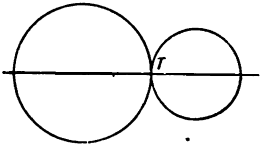


Fig. 96.

But each tooth of A as it passed T took with it a recess of B, so n recesses of B passed, and thus an arc np of B has passed T.

Equal arcs of A and B then pass T in equal times, so the motion is exactly the same as that of pure rolling between the original pulleys. We shall then always represent the toothed wheels simply by two circles, as we did the pulleys. If now

n_A be the number of teeth in A,
n_B " " B,

then, p being the pitch,

$$n_A p = 2\pi r_A, \quad n_B p = 2\pi r_B.$$

[Notice p is round the circumference, not the straight line CE (Fig. 95).]

$$\therefore \frac{r_A}{r_B} = \frac{n_A}{n_B},$$

so that

$$\frac{A_B}{A_A} = \frac{n_A}{n_B},$$

and the angular velocities are inversely as the number of teeth.

We see that A and B must necessarily rotate in opposite directions. But it may be necessary that they

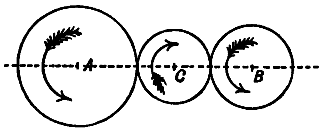


Fig. 97.

rotate in the same direction; to effect this we use an intermediate wheel C.

Looking at the arrows we see B now rotates in the same direction as A. We must see whether this affects the velocity ratio.

Let n_A, n_B, n_C be the numbers of teeth. Then

$$A_B : A_C = n_C : n_B,$$
$$A_C : A_A = n_A : n_C,$$
$$\therefore A_B : A_A = n_A : n_B,$$

so that C makes no difference in the velocity ratio.

We could then insert any number of intermediate wheels as C, each gearing in turn with one before and one behind it, without altering the final velocity ratio,

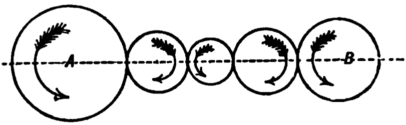


Fig. 98.

and we could thus transmit motion from one shaft to one at some distance.

But we can also so combine wheels as to alter the velocity ratios.

In Fig. 99 there are two intermediate wheels, C and

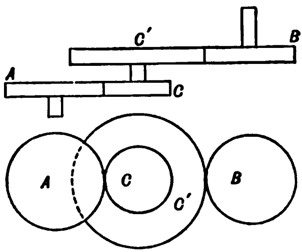


Fig. 99.

C′, both keyed on one shaft—C gearing with A and C′

with B. C and C′ being keyed to the same shaft, we have necessarily

$$A_C = A_{C'}.$$

But

$$A_C = \frac{n_A}{n_C} . A_A, \quad A_{C'} = \frac{n_B}{n_{C'}} . A_B,$$

$$\therefore \frac{n_B}{n_{C'}} . A_B = \frac{n_A}{n_C} . A_A,$$

and

$$\frac{A_B}{A_A} = \frac{n_A}{n_C} \cdot \frac{n_{C'}}{n_B},$$

being the product of the angular velocity ratios of C to A and B to C′.

When we have then a large velocity ratio to obtain we can split it into two factors, each of which is easily obtained.

For example, let a velocity ratio say of 34 to 3 be required, the wheels available being a set from 20 teeth to 120, increasing by 5, *i.e.* 20, 25, 30, 35, etc.

We have no wheels which can give it directly, there being no 34 or multiple of 34; but

$$\frac{34}{3} = \frac{17 \times 2}{3} = \frac{17}{4} \times \frac{8}{3} = \frac{85}{20} \times \frac{80}{30}.$$

We take then A of 85 teeth, C of 20, C′ of 80, and B of 30.

This gives one arrangement, but others could be equally well used, *e.g.*

$$\tfrac{17}{5} \times \tfrac{10}{3} = \tfrac{85}{25} \times \tfrac{100}{30},$$

and so on.

Screw Cutting.—One use of this is in lathes for screw cutting. The saddle bearing the tool is moved by the leading screw; let this have n threads to the inch.

We require to cut now a thread having m to the inch.

The saddle then should move 1 inch, *i.e.* the leading screw turn n times, while the work turns m times, or the lathe mandril turns m times,

$$\therefore \frac{\text{Angular velocity of screw shaft}}{\text{Angular velocity of mandril}} = \frac{n}{m},$$

and we have the problem just considered. If n/m be a small ratio we can obtain it with two wheels only, but then the leading screw and the work would turn in opposite directions, so that a right-handed leading screw would cause a left-handed thread to be cut. To avoid this we should put in an intermediate wheel C, as Fig. 97, simply to change the direction.

If, however, n/m be large, we have to use four wheels in the manner above stated.

[In order to effect the gearing of the wheels a swinging arm is fitted, to which the common axis of C and C′ is fixed.

O_A, O_B are the centres of A and B, A being on the lathe mandril or spindle, B on the end of the leading screw. The swinging arm D swings round O_B.

B being in place the axis of C and C′ is slid along the slot till C′ gears properly with B, then the nut on the axis is tightened up so fixing it to D, then the whole swings round O_B, C′ of course remaining in gear with B, till C gears with A. Then D is fixed in position.]

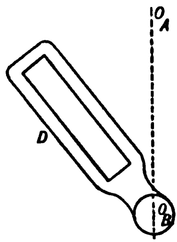


Fig. 100.

Feed Motion of Drill.—In some instances the axes of A and B lie in the same straight line. Here B turns loosely on the spindle of A. This of course does not affect the velocity ratio.

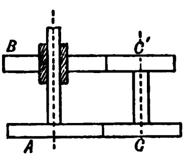


Fig. 101.

If now n_B be slightly greater than n_A, and n_C be slightly greater than $n_{C'}$, A_B will be a little less than A_A, so that A will turn relatively to B with a slow motion, viz. $A_A - A_B$, in the same direction as its own motion.

[In other words, in every second A gains on B an angle $A_A - A_B$.]

This slow relative motion we can utilise as follows :—

In the boss of B cut a thread, and cut on the spindle of A a thread to fit it. B is then a nut and A the screw. Let B be prevented by bearings from rising or falling.

Then as A slowly gains on B the spindle gradually screws out of B and descends. A also would descend with it, and thus the motion would soon stop, for A would leave C. But we can prevent this by connecting A to its spindle, not by a tight key, but by a key which can slide in a long slot in the spindle, and now the spindle can descend leaving A behind in its bearings.

Fig. 102 shows an end view of A and its spindle, *k* the key can slide in the slot, but is fixed to A.

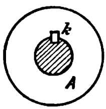


Fig. 102.

In the practical case the spindle of A is that of a drilling machine, and the mechanism becomes a self-feeding arrangement, giving the spindle a slow descending motion as it turns.

Crabs.—The two practical cases we have last considered have been mechanisms, motion being the object aimed at.

As an example in which modification of effort is required we may take the common crab.

This is used in two forms.

In each case the moment is applied to A either by manual effort on a handle or handles, or by an engine.

The resistance is a weight to be lifted, the rope or chain which lifts it passing round a barrel attached to B, thus applying the resisting moment.

If A drive B direct we have a single purchase. If intermediate wheels C, C′ be used we have a double purchase.

We have in the latter case

$$\frac{M_B}{M_A}=\frac{A_A}{A_B}=\frac{n_C}{n_A}\cdot\frac{n_B}{n_{C'}}.$$

A and C′ are small, while C and B are large wheels, hence we obtain a large mechanical advantage.

Bevel Wheels.—When two turning pairs whose axes are not parallel but meet in a point are to be connected, the connection can be effected by wheels as follows:—

Let OC, OD be the given axes. Fix to the shafts two wheels A and B with sloping faces, called Bevel Wheels, so shaped that the lines *cd*, *ab*, and *ef* all meet in O. The wheels are then portions of cones whose apices are at O. Then these two wheels can turn with pure rolling contact.

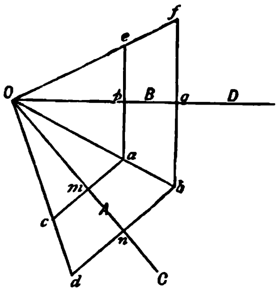


Fig. 103.

To prove this, suppose the contact at *b* is pure rolling, then we will prove it is so also at *a*. Let

$$V = \text{the common velocity of the peripheries at } b,$$

then

$$A_A = \frac{V}{nb}, \quad A_B = \frac{V}{bq}.$$

Now

$$\text{Velocity of periphery of A at the point } a = A_A \times am,$$
$$= V \cdot \frac{am}{nb},$$

and similarly

$$\text{Velocity of periphery of B at the point } a = V \cdot \frac{ap}{bq}.$$

But

$$\frac{am}{nb} = \frac{Oa}{Ob} = \frac{ap}{bq},$$

so that at *a* the peripheral velocities of A and B are identical, and the motion is there pure rolling.

The same proof applies to any point on *ab*, the line of contact. So there is pure rolling everywhere.

By cutting teeth as in the preceding we produce the same relative motion as pure rolling would produce (see page 123), and we then treat the bevel wheels as if they were simply the smooth wheels from which we have derived them. For the velocity ratio we have

$$\begin{aligned} A_B : A_A &= \frac{V}{bq} : \frac{V}{nb} \quad \text{(see preceding work),} \\ &= nb : bq, \\ &= \sin bOC : \sin bOq. \end{aligned}$$

To construct a pair of bevel wheels, or at least the smooth wheels equivalent to them, the faces of which are called the pitch surfaces (compare pitch circle, page 124), we proceed thus—

OC and OD are given, and we can select the maximum or minimum or mean diameter of one wheel.

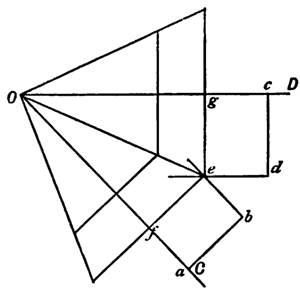


Fig. 104.

Let us take then *ab* as the maximum diameter of A, and set it off perpendicular to OC.

Let n be the given velocity ratio, *i.e.* $A_B : A_A$. Take *cd* equal to ab/n, and set it off perpendicular to OD.

Draw *be*, *de* parallel to OC, OD respectively, meeting in *e*.

Drop *ef*, *eg* perpendicular to OC, OD, and these are the maximum diameters of the two wheels.

We then complete them as shown, taking any convenient thickness. Then

$$\frac{A_B}{A_A}=\frac{ef}{eg}=\frac{ab}{cd}=n, \text{ as required.}$$

The two cones are called the pitch cones, and the angles *e*OC, *e*OD their angles (Fig. 104).

When the angle between the two shafts is 90° the wheels are called Mitre Wheels. If the angle of one cone be 180° the pitch surface is a plane, and the teeth are on the flat side of the wheel; such a wheel is called a Crown Wheel. A crown wheel is usually of large size compared to the bevel wheel which gears with it, and this wheel is called a Pinion. When a small common toothed wheel gears with a large one, the small one is called the Pinion and the large the Spur Wheel.

EXAMPLES.

1. A pair of blocks have three sheaves in the upper and two in the lower block. Find the pull required to raise $\frac{1}{4}$ ton, assuming $\frac{2}{5}$ of the energy wasted. *Ans.* $186\frac{2}{3}$ lbs.

2. Two sheaves 8 ins. and 7 ins. diameter respectively are fastened together, and turn on one axle, forming the upper block of a Weston differential pulley. The lower block contains one sheave only. The left-hand ply of chain supporting the lower pulley passes over the 8 in. diameter to the right, and the right-hand ply over the 7 in. diameter to the left; the two loose ends are then connected, so the chain is endless, hanging in two loops, one loose and one supporting the movable block and weight. Find the weight which can be lifted by a pull of 10 lbs., $\frac{3}{5}$ of the energy being wasted.

Ans. The pull being applied to the loose part hanging from the 8 in. sheave, let it turn the upper pulley once. Then space traversed by pull is 8π ins. The 8 in. sheave winds on from the loop supporting W 8π ins., but the 7 in. sheave must also turn once, being fixed to the 8 in., and this inwinds 7π ins. The tight loop

is thus shortened π ins. (and the loose one lengthened the same amount) so W rises $\frac{\pi}{2}$ ins.

$$\therefore \text{ velocity ratio} = \frac{8\pi}{\frac{\pi}{2}} = 16.$$

$$\text{Whence weight raised} = \tfrac{2}{5} \times 16 \times 10 = 64 \text{ lbs.}$$

3. A shaft is to be driven at 400 revolutions per minute; a pulley on it is 8 ins. diameter. The shaft from which it is to be driven makes 70 revolutions per minute. Find the size of driving pulley necessary. *Ans.* 3 ft. $9\frac{5}{8}$ ins. diameter.

4. In the preceding the belt is $\frac{3}{8}$ in. thick, and the slip is 1 per cent. Allowing for these obtain the correct diameter.
Ans. 4 ft.

5. A pulley 4 ft. diameter is driven by two belts running over each other, each $\frac{3}{4}$ in. thick. The speed of the middle plane of the inner belt is 1800 ft. per minute. How much does the outer gain on the inner per minute? *Ans.* $55\frac{5}{13}$ ft.

6. Calculate the revolutions per minute of a dynamo driven by a belt $\frac{1}{5}$ in. thick, the diameter of the pulley of the dynamo being 6 inches, and the angular velocity of the driving pulley, 4 feet 6 inches diameter, being 10 radians per second. Allow 2 per cent slip. *Ans.* 818.

7. A lathe is to be driven so as, without the use of back gear, to cut brass from $\frac{1}{4}$ in. diameter to 3 in. diameter (see question 7, page 38). The main shop shaft runs at 60 revolutions, and the driving pulley on it is 2 ft. 6 in. diameter; the driving pulley of the lathe is to turn at 40 revolutions. Find the sizes of the overhead driven pulley and of the speed pulleys, the least diameter of the latter being 5 ins., and four speeds are required, the extreme as stated and two convenient intermediate speeds.

Ans. Diameter of driven pulley, 18 ins.
Speed ratios, 10/3, 5/3, 4/5, 10/36.
Diameters of upper pulley, $5\frac{1}{4}$, $8\frac{5}{8}$, $12\frac{7}{8}$, 18 ins.
,, lower pulley, $17\frac{3}{4}$, $14\frac{3}{8}$, $10\frac{1}{8}$, 5 ,,

8. The usual back gear is used in the preceding, viz. a wheel A on the speed pulley drives a wheel C; on the same shaft as C is a second wheel C′ which drives B, which is connected to the mandril and drives the work (see Fig. 101). The speed pulley and mandril are disconnected when the back gear is in use, at other times they are bolted together. A has 18 teeth, and C and B are equal; find the number of teeth in C, C′, and B re-

spectively, so as to reduce the speed of rotation approximately 12 times.

Ans. Since B = C, C′ must equal A, the axes being parallel.

Teeth are therefore in C′ 18, in B and C 63.

9. In the preceding the overhead gear is 12 feet above the lathe, centre to centre. Find the length of a crossed belt for the speed pulleys. *Ans.* 27 ft. 1 in.

10. A belt running at 1500 ft. per minute transmits 80 H. P. Find the difference of tension of the two sides of the belt.
Ans. 1760 lbs.

11. The tension per inch width of a belt must not exceed 110 lbs. Find the width of belt necessary to transmit 12 H. P. from a shaft running at 80 revolutions, the diameter of the driving pulley being 4 ft. 6 ins., and the ratio of the tensions $1\frac{3}{4}$ to 1.
Ans. $7\frac{1}{2}$ ins.

12. The set of wheels for a screw-cutting lathe range from 20 to 150 teeth, there being two 20 wheels. The leading screw has 2 threads to the inch. Arrange suitable trains for cutting threads on, 1st, a $\frac{1}{4}$ in. screw, 20 threads to the inch; 2d, a 1 in. screw, 8 threads to the inch; 3d, a 2 in. screw, $4\frac{1}{2}$ threads to the inch. *Ans.* 1st, $\frac{80}{20} \times \frac{75}{30}$; 2d, $\frac{80}{20}$; 3d, $\frac{90}{40}$.

13. In a single purchase crab the length of the handle is 16 inches and diameter of barrel 8 inches. The pinion on the same axis as the handle has 16 teeth, and the spur wheel connected to the barrel 90 teeth. What weight can one man exerting a push of 30 lbs. lift? *Ans.* 337.5 lbs.

14. The preceding is fitted to act with a double purchase by sliding the pinion out of contact with the spur wheel, and putting in gear a pinion of 18 teeth working with a spur wheel of 54 teeth, on the axis of the latter is another pinion of 18 teeth, which now drives the 90 wheel. Find the force required to lift 1 ton. *Ans.* $74\frac{2}{3}$ lbs.

15. If the pitch of the teeth in the preceding be $1\frac{1}{4}$ inch, find the diameters of the pitch circles. Assuming the mutual pressure between the teeth to act at the pitch point, find its amount in each case.

Ans. 7.16, 21.48, 7.16, 35.8 ins.; 500, $166\frac{2}{3}$ lbs.

16. In the feed motion described on page 128, if the wheel A have m teeth, C n teeth, C′ $n+a$, and B $m+a$, prove that the speed of feed will be $\frac{na}{6n(m+a)}$ Np f.s., where N are the re-

volutions of the spindle per second, and p is the pitch of the screw on the spindle in inches.

17. A friction wheel 4 feet diameter running at 70 revolutions, drives a wheel $2'-3''$ diameter. Find the force with which the wheels must be pressed together per H. P. transmitted, 1st, when metal surfaces meet, coefficient of friction .15; 2d, when the driving wheel is faced with leather, coefficient .4.

Ans. 1st, 250 lbs.; 2d, $93\frac{3}{4}$ lbs.

18. Two shafts meeting at 90° are to be connected with a velocity ratio of 2 to 1. Construct the pitch cones of the bevel wheels. *Ans.* Angles of cones, $63\frac{3}{4}°.26\frac{1}{4}°$.

19. Prove that a straight pinion cannot work correctly with a crown wheel, the axis of which is at 90° to that of the pinion. In what case would the motion be practically accurate?

Ans. All points of pitch surface of the pinion move at the same velocity, but points on the face, *i.e.* the pitch surface of the crown wheel, do not, since those farthest from the centre move quickest, and compare page 121 (friction wheels). The motion is practically correct when the pinion is very thin, since it practically gears only at one point with the crown wheel.

CHAPTER VII

SIMPLE MACHINES WITH FRICTION

THE results obtained in the last chapter will not hold in actual machines, since they neglect the influence of friction. Hence those results were too favourable; but at the same time they are of value as showing a limit which should be approximated to as closely as possible and as a rough guide to what we may actually expect. Also by neglecting the friction at first we are able more easily to grasp the problems to be solved than if we commenced by taking it into account; we thus follow the principle of introducing our difficulties singly if possible, but, at any rate, gradually.

The velocity ratios found are not affected by the question of friction, they being determined geometrically, not statically; but now we cannot proceed, as before, by the Principle of Work, because it takes the form of

$$\text{Energy exerted} = \text{work done} + \text{work wasted},$$

and the work wasted, depending on the pressures between the moving surfaces, will necessitate those pressures being found by the principles of Statics, which in the last chapter we only used for verification.

Also we now introduce a new conception, viz. :—

Efficiency.—When we move a machine, or a pair, then actually there is always work wasted, so that the work done is less than the energy exerted. The ratio

of the work done to the energy exerted is called the Efficiency of the machine, or the pair.

We have then

$$\text{Efficiency} = \frac{\text{work done}}{\text{energy exerted}},$$

or equally, by the Principle of Work,

$$\frac{\text{Work done}}{\text{work done} + \text{work wasted}}, \text{ or } \frac{\text{Energy exerted} - \text{work wasted}}{\text{energy exerted}}.$$

We use whichever is the most convenient to calculate.

Sometimes we need a term to represent the reciprocal of the efficiency, and this we call the **Counter Efficiency.**

$$\begin{aligned} \therefore \text{ Counter-efficiency} &= \frac{\text{work done} + \text{work wasted}}{\text{work done}}, \\ &= 1 + \frac{\text{work wasted}}{\text{work done}}, \\ &= 1 + e, \end{aligned}$$

the letter e being often used for the purpose here shown, of denoting the ratio of work wasted to work done.

We have

$$\begin{aligned} \text{Efficiency} &= \frac{\text{work done}}{\text{energy exerted}}, \\ &= \frac{\text{resistance} \times \text{distance it moves}}{\text{effort} \times \text{distance it moves}}, \\ &= \text{force ratio} \times \text{velocity ratio}, \end{aligned}$$

so that we see that we cannot now determine one of these from the other without knowing the efficiency. Since the efficiency is necessarily less than unity, it follows that for a given velocity ratio the force ratio is less than it would be if there were no friction (page 96), so that the mechanical advantage is decreased by the friction.

[The term efficiency is sometimes used with very different meanings, and it is necessary to be very careful to find out, in

any given case, exactly what is meant by it. This remark applies generally, and not especially to our present subject. What we mean by efficiency has been already stated, and in that sense only will the term be used.]

We will now commence with the

Inclined Plane.

Comparing with Fig. 72, page 98, we have an extra force acting, because the action of the plane is now a double one, although there are still only the same number of bodies acting. We see then that in answering the questions on page 97 we must be careful to see that we take account of the *total* action of each body.

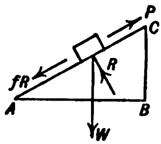


Fig. 105.

The extra force is fR, where f is the coefficient of friction (page 51). Hence we must determine R by Statics. This gives, since fR and P have no effect in the direction of R, the effect of W in that direction equal to the effect of R, *i.e.*

$$W \cos CAB = R,$$

as on page 99, the extra force fR not altering the equation,

$$\therefore fR = fW \cos CAB.$$

Then

$$\text{Energy exerted} = P \,.\, AC,$$

taking the movement from A to C.

And here we may introduce another caution, in addition to those of page 97, viz. that we must clearly define in our minds exactly the period during which we intend to apply the equation or principle of work. The period now taken is from the instant of starting from A to that of arriving at C.

Resuming, we have

$$\text{Work done} = W \times CB,$$

and

$$\begin{aligned} \text{Work wasted} &= fR \times AC, \\ &= fW \times AC \cos CAB, \\ &= f \,.\, W \times AB. \end{aligned}$$

And

$$\therefore \; P . AC = W . BC + fW . AB.$$

The force ratio is now

$$\frac{W}{P} = \frac{AC}{BC + f . AB},$$

being, as we have already stated it must be, less than the reciprocal of the velocity ratio.

The foregoing work applies certainly to the case where W actually slides on the plane, but does it now apply to the pseudo sliding of a wheeled vehicle?

We have already stated that such resistance does, on a horizontal plane, practically follow the laws of sliding friction; but at the same time we know that actually the resistance arises from the axle friction, and also from the peculiar action between the rolling wheels and the ground, for an explanation of which we must refer to the larger treatise. It might seem then at first sight that the amount of resistance would be, not fR, but fW, because the carriage still rests on its axles as it does on a horizontal plane. But further consideration shows that the whole weight does *not* now rest on the axles, because the pull P tends to lift the carriage off them; and it does this exactly in the same proportion as it would relieve the plane of a part of the weight of a sliding piece, and hence the friction is reduced in the proportion of R to W, *i.e.* it is fR or fW cos CAB as for a slider.

In ordinary cases of gentle inclines it is not, however, practically necessary to consider this, because cos CAB approaches so nearly to unity. Thus on a slope of 1 vertical to 50 along the road

$$\cos CAB = \frac{AB}{AC} = \frac{\sqrt{2500 - 1}}{50}.$$

We have treated motion up the plane, but we have also motion down the plane to consider.

In this case P is down the plane and fR up. Also W is now an effort as well as P. Then

$$\text{Energy exerted} = \text{P} \,.\, \text{AC} + \text{W} \,.\, \text{BC},$$
$$\text{Work done} = 0,$$
$$\text{Work wasted} = f \,.\, \text{R} \,.\, \text{AC},$$
$$= f\text{W} \,.\, \text{AB, as before.}$$

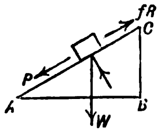


Fig. 106.

There is no *work done* in the sense of useful or recoverable work, such as the lifting up of the weight, the energy exerted in which can be recovered by allowing it to fall again; so that the efficiency is zero. But we must not take this to mean that there is no useful effect, because we wish the carriage or train to go down the slope, and so the effect produced is what we required. We must, however, be consistent, and so we say all the work is waste work.

The equation of work is then

$$\text{P} \,.\, \text{AC} + \text{W} \,.\, \text{BC} = f\text{W} \,.\, \text{AB},$$
$$\therefore \text{P} = \frac{\text{W}(f \,.\, \text{AB} - \text{BC})}{\text{AC}},$$

and hence may be either positive, zero, or negative, as

$$f \,.\, \text{AB is} > = \text{or} < \text{BC}.$$

The meaning of this is, that, taking say a train—

In the first case, to keep the motion uniform the engine must exert a pull.

In the second, the train will just run uniformly by itself.

And in the third, there must be a backward push; this would not, however, be applied as an actual push, but by applying the brakes, f would be increased till f. AB became equal to BC.

When we have

$$f \,.\, \text{AB} = \text{BC},$$

then

$$f = \frac{\text{BC}}{\text{AB}} = \tan \text{CAB}.$$

And we have thus a means of determining, experi-

mentally, the coefficient of friction between a pair of surfaces. To use this method we make one an adjustable inclined plane, and the other a slider; and then the coefficient is the tangent of the angle at which the plane slopes, when the slider, *being set in motion*, will continue sliding uniformly down it.

On page 138 we have the equation

$$P \,.\, AC = W \,.\, BC + fW \,.\, AB.$$

Now W . BC is the work which would be done in lifting W direct from B to C, and fW . AB is the work which would be done against friction in drawing W along AB, supposing AB to have the same roughness as AC. It follows then that—

The energy required to draw a load up an inclined plane is equal to that required to draw it along the equally rough base and lift it through the height.

If then ADC (Fig. 107) were another path we could go along it with the same expenditure of energy as along AC.

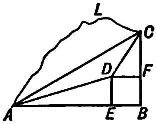


Fig. 107.

For the energy required would be to drag along AE and DF, which together equal AB; and to lift through DE and FC, together equal to BC.

And we can extend this reasoning to any extent, and say finally that the energy required is independent of the path chosen. The path may even rise above C as ALC, the principle still holds, because gravity would restore during the descent the energy it had expended on it during the ascent above C. The preceding of course requires that we allow gravity full play during the descent; if, as is usual, a brake be applied during descent, then the principle no longer holds.

We have just seen that a truck running down an inclined plane, the tangent of the inclination being greater than f, requires a pull back to keep its motion uniform, or else an increased frictional resistance. If

we use the latter we waste work, but we can in some cases use a pull, which being transmitted by a rope can be used as an effort for some useful purpose.

Double Inclined Plane.—A practical example is found in the carriage of stone from an elevated quarry down to a lower level, as that of the quay at which vessels are loaded. We then use two lines of rails side by side, on one inclined plane, the side and front view being here shown.

A loaded truck running down exerts a pull on the

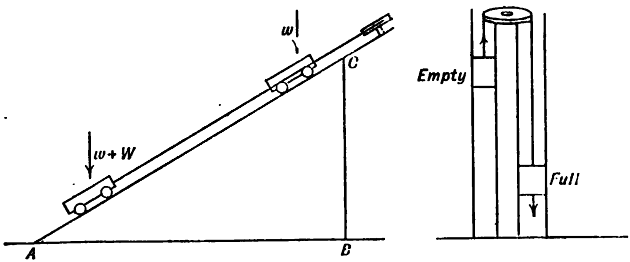


Fig. 108.

tail rope, which is led round a pulley at the top, and then down the other line and fastened to an empty truck coming up. Let now

$$P = \text{pull on rope},$$
$$w = \text{weight of a truck},$$
$$W = \text{load carried}.$$

Then for uniform motion—
of the empty truck,

$$P \,.\, AC = w \,.\, BC + f \,.\, w \,.\, AB\,;$$

of the full truck,

$$(W + w)BC = P \,.\, AC + f(W + w)AB.$$

These are the equations of work, and hence we obtain

$$f(2w + W)AB = W \,.\, BC.$$

The same result can be obtained by considering the machine as a whole.

In this case

$$\text{Energy exerted} = W \,.\, BC,$$

because the ascent and descent of the w's balance each other;

$$\text{Work done} = 0,$$
$$\text{Work wasted} = f(W + w)AB + fw \,.\, AB.$$

Whence at once

$$W \,.\, BC = f(W + 2w)AB.$$

[We have here a good illustration of the character of friction always resisting the motion, whether up or down.]

There is then for any given load a certain slope down which the trucks will steadily run, given by

$$\tan CAB = \frac{BC}{AB} = f \,.\, \frac{W + 2w}{W}.$$

If, as in some cases, the slope is greater than this; then a frictional force must be applied to the top pulley to keep the motion steady. This has the effect of increasing the pull on the load side and decreasing it an equal amount on the light side, similar to a belt (page 120). Let

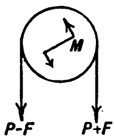


Fig. 109.

M = friction moment applied to the pulley,
$P + F$ = pull of loaded side of rope,
$P - F$ = ,, light ,, ,,
r = radius of pulley.

Then, considering the equilibrium of the pulley,

$$M = (P + F)r - (P - F)r,$$
$$= 2Fr.$$

being a couple F, F, on an arm $2r$.

We can now find F, which will give us the required M.

For we have, for loaded truck,

$$(W + w)\, BC = (P + F)\, AC + f(W + w)\, AB\,;$$

for light truck,

$$(P-F)\,AC = w\,.\,BC + f\,.\,w\,.\,AB.$$

Whence

$$(W+w)\,BC - F\,.\,AC = F\,.\,AC + w\,.\,BC + f\,.\,AB(W+2w),$$
$$\therefore\ 2F\,.\,AC = W\,.\,BC - f\,.\,AB\,(W+2w),$$

which gives us the value required.

[The student should obtain for himself this result by considering the machine as a whole (see preceding case).]

If we wish to draw up heavy trucks then the pulley at the top must be connected to an engine which forces it to revolve against the difference of tension. In this case, the empty trucks being lowered render a less effort on the part of the engine necessary. Taking this case as a whole, we have

Energy exerted by engine $=$ M $\times$ angle turned through by pulley.

But the angle turned through by the pulley is, in circular measure, AC/r, and $M = 2Fr$.

$$\therefore\ \text{Energy exerted} = 2Fr \times \frac{AC}{r} = 2F \times AC,$$

$$\text{Work done} = W\,.\,BC.$$
$$\text{Work wasted} = f(W+w)AB + f\,.\,w\,.\,AB.$$
$$\therefore\ 2F\,.\,AC = W\,.\,BC + f(W+2w)AB.$$
$$\therefore\ M = 2F\,.\,r = \frac{W\,.\,BC + f(W+2w)AB}{AC}\,.\,r.$$

We may now find the necessary power of the engine to effect the lift in a given time.

For let time of lift be t minutes, W etc. be in lbs., and AB etc. in feet. Then

$$\text{Energy exerted in } t \text{ minutes} = W\,.\,BC + f(W+2w)AB \text{ ft.-lbs.}$$
$$\therefore\ \text{Energy exerted in 1 minute} = \frac{W\,.\,BC + f(W+2w)AB}{t} \text{ ft.-lbs.}$$

and

$$\therefore\ \text{Horse Power} = \frac{W\,.\,BC + f(W+2w)AB}{t \times 33000}.$$

[Notice that the last expression is a number simply, not foot-lbs. Suppose, for instance, it were 6, then the engine is equivalent to six horses, not six ft.-lbs. horses, which is evidently nonsense, yet this is a common mistake.]

Practical examples of all the preceding cases can be found in cable tramways, in which, at one and the same time, some cars may be running on a level, others going up inclines, and others going down; and the weights vary as they are more or less occupied.

The Wheel and Axle.—We have seen that the friction of a turning pair depends on the pressures on the bearings and their radii (page 69).

The principal part of the problem then will be the determination of these pressures.

1st Case.—The **Wheel and Axle** in its usual ideal form. Let

$$R = \text{radius of wheel,}$$
$$r = \quad ,, \quad \text{axle,}$$
$$r' = \quad ,, \quad \text{bearings.}$$

If the bearings were of different diameters it would

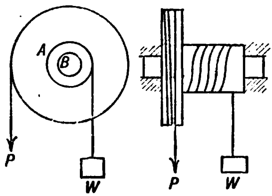


Fig. 110.

be necessary to consider them separately, and hence find the pressure on each. We will, however, suppose them equal, and we can then treat them as one.

We have then

$$\text{Total pressure} = P + W,$$
$$\therefore \text{ Moment of friction} = f(P + W)r' \quad \text{(page 69).}$$

And hence for one revolution

$$P \times 2\pi R = W \times 2\pi r + f(P + W)r' \times 2\pi,$$

or

$$PR = Wr + f(P+W)r',$$

$$\therefore \text{ Force ratio} = \frac{W}{P} = \frac{R - fr'}{r + fr'},$$

and

$$\text{Efficiency} = \text{Force ratio} \times \text{velocity ratio},$$

$$= \frac{R - fr'}{r + fr'} \times \frac{r}{R},$$

which is independent of W.

It will not be so, however, if we consider the weight of the wheel and axle.

2d Case—**The Windlass.**—We will not take any particular dimensions, because, as we shall now see, we cannot obtain in any simple manner results of practical value.

Consider how the effort is applied, actually it will be continually varying both in magnitude and direction, but let us suppose it applied in a definite manner, say of constant magnitude, and always at right angles to the handle.

The friction of each bearing depends on the total pressure on it, and that on the two bearings on the *total* pressure on the two; this total pressure also happened in the last case to be the *resultant* pressure P + W, and hence was easily found, and constant. But now we shall have to find the actual pressure on each, and this will be continually varying as the direction of P varies, so that we should have to find it for every position of the handle. Then for each position we could find the friction moment for each bearing. These being variable, the work wasted would be found graphically by drawing a curve of moment, and finding its area (page 80).

Such a process as the foregoing is rarely if ever necessary, and it is therefore sufficient to indicate it simply, without entering into the details.

A similar remark applies to the capstan.

The Screw.—The friction we here consider is that of the screw pair alone, if there be other pairs their frictions must be calculated separately.

Let then

M = turning moment.
W = weight lifted or resistance to motion.
l = length of thread in nut.
d = mean diameter of screw (*i.e.* midway between top and bottom of thread).
f = coefficient of friction between screw and nut.
p = pitch of thread.

The screw is then acted on by W, M, the normal pressure between its thread and the nut, and the friction also between these. The two latter forces are distributed over the whole length l, and across the breadth of the thread, and we shall treat them as if they were all concentrated at the mid-breadth of the thread, so that the screw reduces to the form here shown, the thread being represented by the single line on a cylinder of diameter d or radius r, these being the mean diameter and mean radius of the screw.

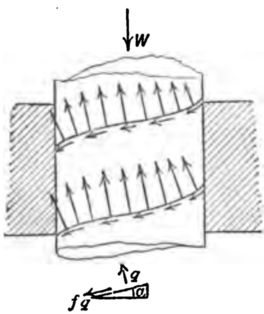


Fig. 111.

The distributed forces are represented by the small arrows. These appear to be in varying directions, but this is due to our point of view. When we look straight at the screw, the small piece directly in front of us is at an angle α to the horizontal such that $\tan \alpha = p/\pi d$ (page 109), and this is true wherever we stand.

If now we resolve all the forces vertically, we have, if R = sum of all the small normal pressures,

Vertical resultant of all the normal pressures $= \mathrm{R} \cos \alpha$.

[See the small figure which represents one small force, and is the same for each of them.]

Vertical resultant of all the small frictional forces $= f\mathrm{R}\sin\alpha$.

[See small figure again.]

Then $\mathrm{R}\cos\alpha$ upwards balances W and $f\mathrm{R}\sin\alpha$ downwards, M having no vertical effect,

$$\therefore\ \mathrm{W}+f\mathrm{R}\sin\alpha=\mathrm{R}\cos\alpha,$$

or

$$\mathrm{W}=\mathrm{R}(\cos\alpha-f\sin\alpha).$$

Also each small force, call it q say, has a horizontal component $q\sin\alpha$.

And the small frictional force fq has a horizontal component $fq\cos\alpha$.

These act in the same direction, and have together a turning moment

$$(q\sin\alpha+fq\cos\alpha)r$$

about the axis of the screw.

The same is true for each small force,

$$\therefore\ \text{Total moment}=(\mathrm{R}\sin\alpha+f\mathrm{R}\cos\alpha)r.$$

[Since all the q's together make up R.]

The nut then offers this resisting moment to the turning of the screw, and there being no other body resisting the turning, we have

$$\therefore\ \mathrm{M}=(\mathrm{R}\sin\alpha+f\mathrm{R}\cos\alpha)r.$$

Taking then one revolution

$$\begin{aligned}\text{Energy exerted}&=\mathrm{M}\times 2\pi,\\&=2\pi r\mathrm{R}(\sin\alpha+f\cos\alpha).\\ \text{Work done}&=\mathrm{W}\times p,\\&=\mathrm{R}(\cos\alpha-f\sin\alpha)p,\\ \therefore\ \text{Efficiency}&=\frac{p(\cos\alpha-f\sin\alpha)}{2\pi r(\sin\alpha+f\cos\alpha)}.\end{aligned}$$

But

$$\frac{p}{2\pi r}=\tan\alpha.$$

$$\therefore \text{ Efficiency} = \frac{\tan\alpha(\cos\alpha - f\sin\alpha)}{\sin\alpha + f\cos\alpha},$$
$$= \frac{1 - f\tan\alpha}{1 + f\operatorname{cotan}\alpha}.$$

Also we can write f as the tangent of an angle, say ϕ (page 140),

$$\therefore \text{ Efficiency} = \frac{1 - \tan\phi \, . \tan\alpha}{1 + \tan\phi \, . \operatorname{cotan}\alpha},$$
$$= \frac{\tan\alpha}{\tan(\alpha + \phi)},$$

A very simple form, which can be easily remembered, and we must, of course, also remember what ϕ is.

Pulleys.—There would not in the pulleys be much difficulty in determining the axle friction of the sheaves; but the chief frictional resistance does not arise from this, but from the resistance of the ropes to being bent and unbent, arising from an internal friction between the fibres. This also would appear in the wheel and axle.

The kind of friction just spoken of is outside our present limits, and is, in fact, not known with any great accuracy. Hence we cannot effect a detailed examination in this case, but we can, however, treat the friction as a whole in the following manner:—

Take any system of pulleys and experiment with different loads, finding in each case the effort required to steadily lift the load. Let

$$\begin{aligned} P &= \text{effort}, \\ W &= \text{load}, \\ y &= \text{distance P moves}, \\ x &= \text{distance W moves}. \end{aligned}$$

x/y is then the velocity ratio, and y/x would be the mechanical advantage if there were no friction, since we should have

$$Py = Wx.$$

But now we shall find that all our experimental results will satisfy a formula or equation

$$Py = Wx + eWx + P_0y,$$

where e and P_0 are constant for all values of W and of P.

This is said to be the law of efficiency of lifting tackles or pulley systems. Two experiments will be sufficient to determine the values of e and P_0. One may be with $W = 0$, *i.e.* no weight at all; then we have $P = P_0$ (from above), so that P_0 is the effort required to bend and unbend the ropes, and to turn the sheaves on the axles against the friction caused by the weights of the block and ropes. The remaining part eWx of the waste work is the loss due to pressures caused by the actual load lifted.

There appears then to be a separation into two distinct kinds of waste, but we cannot separate them perfectly because they interact on each other. The equation or law given is on the whole fairly accurate.

The efficiency is

$$\frac{Wx}{Py} = \frac{Wx}{Wx(1+e) + P_0y},$$

and the counter efficiency $= 1 + e + P_0y/Wx$, so that if W be a very large weight compared with P_0, $1 + e$ is practically the counter efficiency, so we could roughly determine $1 + e$ by lifting a very large load and measuring the value of the counter efficiency Py/Wx.

We have dealt pretty fully with the cases we have so far considered, as they exhibit the methods we must use in the great majority of machines. Space prevents our examining any more of the simple mechanisms of chap. v, but the method we have used will apply.

EXAMPLES.

1. A tram car weighing 4 tons, resistance on the level 15 lbs. per ton, is pulled up an incline of 1 in 18. Find the pull required, and also the amount of error which would be made by taking the resistance to be the same on the slope as on a level.

Ans. 557 lbs.; .114 lbs.

2. If 8 passengers, each weighing 130 lbs., enter the car, what increase of slope is this equivalent to?

Ans. 1 in 1200 about.

3. The draught of a waggon is 40 lbs. per ton, and the coefficient of friction between the skid and road $\frac{1}{4}$. In going down an incline, with one of the four wheels skidded, the speed is the same as on a level road with no skid, the horses exerting the same pull. Find the slope. *Ans.* 1 in $17\frac{1}{2}$.

4. The engine of a goods train can just pull it on a level at 25 miles per hour. Resistance 17 lbs. per ton. Two points A and B on the line are 10 miles apart, and B is 60 ft. below A; the slope from A to C, an intermediate point, is 1 in 300, and from C to B 1 in 600. Find at what point of the down grade the speed which was 25 miles per hour passing A will again be the same, and what was the speed at the summit. Also what reduction of power must then be made to keep the speed from rising.

Ans. 2.12 miles from the summit, $8\frac{1}{8}$ miles per hour. Reduce by .22 of original power.

5. A locomotive weighs 45 tons, of which .48 rests on the driving wheels. What must be the coefficient of friction between the surfaces of the driving wheels and the rails that the engine may just draw a train, total weight 200 tons, at 50 miles per hour without slipping, up an incline of 1 in 300. Resistance 45 lbs. per ton. (This friction is called the adhesion.)

Ans. .217.

6. A ship weighing 2000 tons is launched. Find what slope of the ways is necessary for uniform motion when once started. Also, what should be the area of bearing surface so that the pressure shall not exceed $2\frac{1}{2}$ tons per sq. ft., and so force out the tallow? Coefficient .14. *Ans.* 8°; 800 sq. ft.

7. The trucks of a double incline weigh 4 tons, and are loaded with 5 tons. Find the slope so that the loaded truck would run steadily down. Resistance 17 lbs. per ton. *Ans.* 1° 9′.

8. The actual slope in the preceding being 30°, find what frictional moment must be applied to a pulley 8 ft. diameter to keep the motion uniform. *Ans.* $9\frac{3}{8}$ ft.-tons.

9. If the bearings of the pulley be 6 ins. diameter, and the friction coefficient $\frac{1}{18}$, solve the preceding, taking account of the friction of these bearings. *Ans.* 9.57 ft.-tons.

10. The wheels of a railway carriage are 3 ft. 6 ins., weight 18 tons. The coefficient of friction between the brake blocks and wheels is .4. Find the total pressure between the blocks and wheels, so that if detached on an incline of 1 in 120 the

carriage may not run down. Ordinary resistance, 14 lbs. per ton.
Ans. 210 lbs.

11. In question 2, page 131, the diameter of the axles of the pulleys is $\frac{3}{4}$ in. Coefficient of friction $\frac{1}{12}$. Find the efficiency, omitting stiffness of the ropes. *Ans.* .88.

12. In question 6, page 110, the mean diameter of the thrust rings is 15 ins. Coefficient of friction .05. Find the efficiency of the thrust block—shaft pair.

Ans. Work done per revolution, $120 \times 2\pi$ inch-tons. Work wasted, 10.56 inch-tons.

$\therefore$ Efficiency $= .986$.

13. In question 7, page 91, the bearings are 7 ins. diameter. Coefficient $\frac{1}{18}$. Draw a curve of frictional moment and calculate the work lost in friction per lift.

Ans. Total weight is constant, therefore curve is a straight line. Work lost $= 5.9$ ft.-tons.

14. In question 11, page 133, calculate the efficiency, if the diameter of bearings for each shaft be $3\frac{1}{2}$ ins., and the driven pulley 2 ft. 3 ins. diameter. Coefficient $\frac{1}{16}$. *Ans.* .939.

15. Find the efficiency of the screw jack of question 9, page 38. Coefficient .06. Depth of thread $\frac{18}{40}$ of the pitch. Diameter 3 in. *Ans.* .45.

16. Find the law of efficiency of a pair of three-sheaved blocks in which a 12 lbs. pull raises 40 lbs., and a 70 lbs. pull 300 lbs.
Ans. $P = .223W + 3\frac{1}{13}$ in lbs.

CHAPTER VIII

THE DIRECT ACTING ENGINE—MOTION

OF all machines the above is probably the most important, and hence we will examine it thoroughly in detail, so far as the limits of the present work will allow.

Fig. 112 shows the construction of a vertical engine

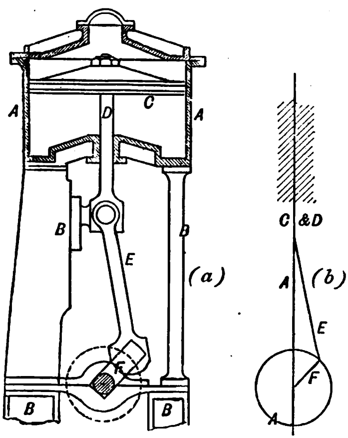


Fig. 112.

for marine propulsion, and will serve as a type of all direct actors.

A is the cylinder; B, B, B the framing; C the

piston; D the piston rod; E the connecting rod; and F the crank shaft.

The shape or construction of the parts is not, to us, of importance except in so far as it governs the motion, *i.e.* we care only about the positions and shapes of the bearing surfaces.

We have then as essentials—

1st. A fixed piece A and B, B, B. This we consider as only one piece, because, although it is actually made in parts, this is only for constructive reasons, and as far as the motion is concerned it might be one solid casting.

[In some small engines it is actually so.]

2d. A piece, C and D, which slides in A (see chap. i. page 19).

3d. A piece E, which is connected to D by a pin joint, so that E turns relatively to D. The centre of the pin is in the centre line of C and D.

4th. A piece F, which can turn in the end of E, and also in a bearing in B, *i.e.* in A. So we have returned to A again. The centre line of the last bearing, or bearings, must meet the line of stroke, as shown at O in the skeleton figure.

The machine consists then of four pieces, connected as here shown:—

Sliding Pair	Framing and cylinder	
	Piston and rod	
Turning Pair		Turning Pair.
	Connecting rod	
Turning Pair	Crank shaft	

The *kind* of relative motion of these pieces we have given us by the connections, and we now wish to find the relations which exist between their *amounts*.

For this purpose we do not need the outlines and cross dimensions as given in Fig. 112 (*a*), but simply the dimensions of the skeleton figure (*b*). In (*b*) the framing

is represented by the paper and the pieces by their centre lines only. The small part shaded is to show that the piece C and D slides in the direction of its length. The dotted circle shows the path of the centre of the crank pin, generally called the Crank Pin Circle.

Position of Piston.—We will first consider how to find the position of the piston when we know that of the crank and *vice versa.*

First we must notice that, since the piston rod and crosshead all slide as one body in a straight line, it follows that if we know the position of any one point in this body, then the position of the whole is fully determined. There is then no need to do more than find the

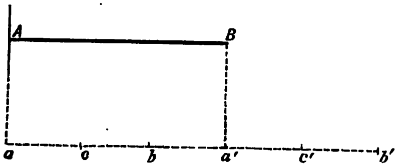


Fig. 113.

position of this one point, and then the position of all the others follow.

Suppose, for example, we select the centre of the crosshead bearing, which in the skeleton figure we should call the end of the piston rod. Then in Fig. 113 we have drawn the piston and rod, A being the piston and B the centre of the crosshead—the dotted line below is drawn to show the travel on.

Take ab to represent the stroke of the piston. Then when

A is at a, B is at a' (beginning of its travel),
A is at b (end of its travel), B is at b' (end of its travel),
A is at c (centre of its travel), B is at c' (centre of its travel),

and generally when

A is distant x from c, B is distant x from c'.

These results .depend on the fact that AB is a constant length, *i.e.* length of skeleton rod, and they show that B travels in its path exactly as A travels in its path.

Our problem then is reduced to finding the position of the end of the piston rod for a given crank position.

In Fig. 114 O is the centre of the crank circle, AA′ its diameter in the line of stroke; then A and A′ are called the *dead points* or *dead centres*, and AA′ is called the line of dead centres. Let

$$a = \text{length of crank arm,}$$
$$na = \text{length of connecting rod.}$$

Take now OP representing a given position of the crank.

Then producing A′A the end of the piston rod must

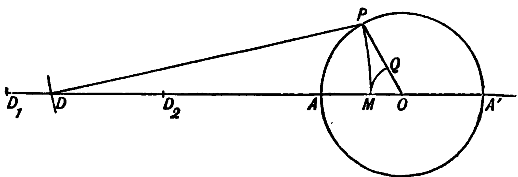


Fig. 114.

lie on this line, since it is the line of stroke. But the end of the rod must also be at a certain distance from P, because it is connected to P by the connecting rod. If then D be the position of the end of the rod, PD must be the length of connecting rod. We can then at once see how to find D, for taking a radius na, sweep out an arc with P as centre, and the point D where this arc cuts A′A produced is the position of the piston rod end.

Take now AD_1 and $A'D_2$, each equal to na. Then D_1 and D_2 represent the ends of the stroke, and the point D moves forward and backward along D_1D_2, showing the position of the piston rod end in its stroke. But looking back we see that the piston moves identically in its stroke, and so we can say D_1D_2 shows the path of the piston, and D gives the piston position.

But we can still further extend this method: for AA′ is equal to D_1D_2, so that AA′ may be taken to represent the piston stroke.

Now with centre D, radius DP, describe the arc PM, cutting AA′ in M.

Then $DM = na$, and so as D moves along D_1D_2, M will move identically along AA′.

It follows then that the position of M in AA′ is identical with that of the piston in its stroke, and hence we say M gives us the piston position. We see then now how to find the piston position corresponding to a given crank position, and we have seen that it may be represented in two, or even more if we like, different ways.

Graphic Representation.—When we have obtained a piston position corresponding to a given crank position, or *vice versa*, then by marking down in some way the result of our construction, we shall be able at any future time to obtain this particular result without the trouble of going again through the construction. Now the simplest way of defining the piston position is by its distance from the middle of its stroke, *i.e.* (Fig. 114) the distance OM.

Suppose then that along OP, the crank position, we set off $OQ = OM$, then we could rub out the whole of our construction, leaving only the point Q, and if at any time we required the position of the piston corresponding to the position OP of the crank, we should only require to measure OQ, and we should then know it was this distance from the middle of its stroke.

Repeat the above construction now for a large number of crank positions OP_1, OP_2, etc. obtaining the points Q_1, Q_2, etc. (Fig. 115). Now for clearness transfer the Q's to a new figure (Fig. 116).

By means of the figure thus obtained we could find the piston position at any time, for any one of the particular cranks OP_1, OP_2, etc. marked on it.

But now, instead of leaving simply a set of discon-

nected points, Q_1, Q_2, etc. draw a continuous fair curve

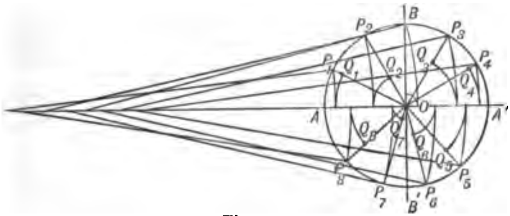


Fig. 115.

through them as completed in Fig. 116. Now we can determine the piston position for any crank whatever. For draw any crank, as Op, then Oq where Op meets the curve is the distance of the piston from the middle of its stroke, when the crank lies along Op.

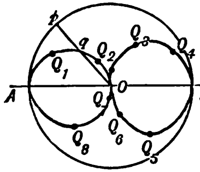


Fig. 116.

The curve which we have just drawn is called the curve of piston position, and it may be of advantage now to summarise the method of constructing it; our explanation so far being more directed to showing how we are naturally led to construct such a curve than to indicate how we should practically set to work on it.

We proceed then thus (Fig. 117):—

Draw the crank circle on any convenient scale. Draw the line of dead centres or stroke, and produce it. Divide the crank circle into a large number of parts, say 24, commencing at A, and number them 0, 1, 2, etc., up to 23. Set a pair of compasses to the length of the connecting rod, on the same scale as the crank circle is drawn, and from 0 describe a circle cutting the line of stroke in 0, from 1 an arc cutting the line in 1, and so on up to 12 or A′. There is no need to go farther, as we shall only get the same points over again, so simply number them back again.

Shift now the needle point to the points on the line

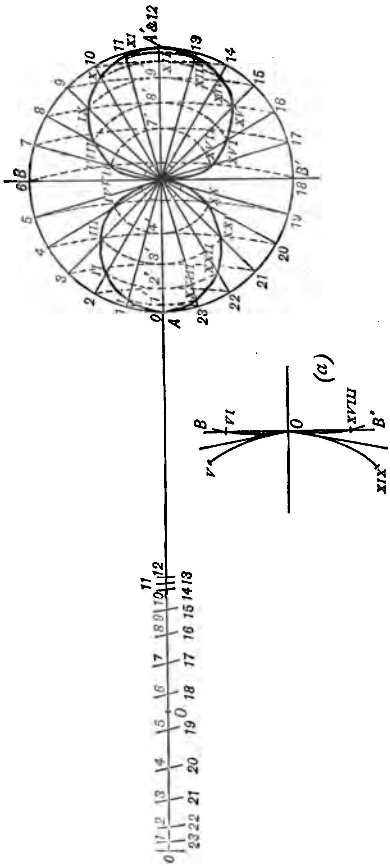


Fig. 117.

of stroke, and describe successively the arcs 11′, 22′, 33′, etc.

Draw the cranks O1, O2, O3, etc., and with fixed centre O draw arcs—

1′I, cutting O1 in I.
2′II, cutting O2 in II.
. . . .
23′XXIII, cutting O23 in XXIII.

Now draw a fair curve through A, I, II . . . O . . . XXIII, A.

The process can be followed in the figure; and examples should be drawn to as large a scale as possible.

The part of the curve near O is not very clear in the main figure, so it is shown enlarged in Fig. 117 (*a*). Here we see that there are, we may say, two separate enclosed curves, having as common tangents the cranks corresponding to zero distance of the piston from the centre of its stroke, *i.e.* piston at O, taking AA′ as the stroke. To find these crank positions we simply reverse our original construction for piston position, *i.e.* from O′, the centre of 0, 12 the stroke, we strike an arc with radius O′O, *i.e.* connecting rod length, cutting the crank circle in two points O_1 and O_2 (Fig. 118), and then OO_1 and OO_2 are the two required cranks. The curve which lies to the left of these cranks shows distances to the left of the centre, and that lying to the right distances to the right of the centre. For cranks very near OO_1 or OO_2 it is not advisable to trust to the curve but to perform the original construction.

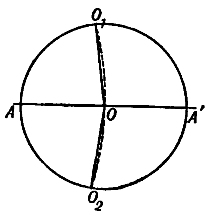


Fig. 118.

We can now use this curve to give us the piston position corresponding to a given crank, or *vice versa*. The former we have already considered, for the latter we proceed thus:—

Let the given piston position be at a distance x from its mid-position.

Take a radius x and with centre O describe a circle. This circle cuts the curve in four points, either of which joined to O gives a crank position fulfilling the required condition. To determine which is the particular one we must know on which side of the centre the piston is, and in which direction it is travelling.

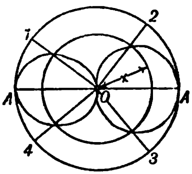


Fig. 119.

Thus in Fig. 119 if the piston be

In OA	travelling	to A′,	O1 is the crank	
,, ,,	,,	from A′,	O4	,,
,, OA′	,,	,, A′,	O3	,,
,, ,,	,,	to A′,	O2	,,

Obliquity of Connecting Rod.—We have seen that the piston position corresponding to a given crank position depends on the ratio of connecting rod to crank arm.

Now to see in what way an alteration of connecting

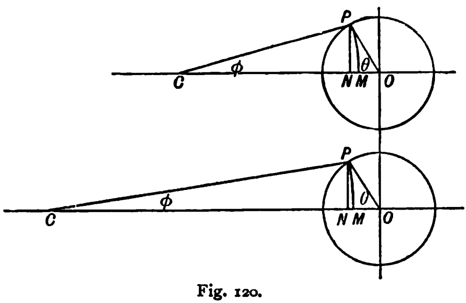


Fig. 120.

rod length does affect the position, let us examine Fig. 120, where we have two examples drawn, differing only in length of connecting rod.

In each figure O is the crank centre, OP a given crank, CP the connecting rod, and M the piston position, PM then being an arc with centre C.

In each figure drop PN perpendicular to the line of stroke.

In each figure the distance CN is less than CM or CP, the connecting rod length. The amount of the difference in each case depends on the angle PCO, *i.e.* on the *obliquity* of the connecting rod; and it varies from zero at the beginning and end of the stroke to a maximum somewhere between.

Contrasting the two figures we see that MN is much less in (*b*) than in (*a*), due to the fact that the connecting rod is longer in (*b*). If we made the connecting rod still longer, the difference MN would be less still; and when the ratio of connecting rod to crank reaches say about 12 : 1, the points M, N would be practically identical.

The process of dropping a perpendicular from P is a simpler one than that of finding the arc PM; and, moreover, the result obtained can be expressed in a simpler form. Hence, for many purposes, we treat the point N as the piston position instead of M; our error in doing so is MN; and we say MN is the *error due to obliquity*, and N is the piston position *neglecting obliquity*.

We can very easily calculate the amount of the error due to obliquity. For with the usual lettering (Fig. 120)

$$\begin{aligned} \mathrm{MN} &= \mathrm{CM} - \mathrm{CN}, \\ &= na - na \cos \mathrm{PCN}. \end{aligned}$$

PCN is usually denoted by ϕ and PON by θ,

$$\begin{aligned} \therefore \ \mathrm{MN} &= na\,(1 - \cos\phi), \\ &= 2\,na \sin^2 \frac{\phi}{2}. \end{aligned}$$

The greatest value of ϕ is when the crank is upright, *i.e.* $\theta = 90°$; and then

$$\sin\phi = \frac{a}{na} = \frac{1}{n}.$$

It is quite near enough, ϕ being a small angle, to take

$$\sin\frac{\phi}{2} = \tfrac{1}{2}\sin\phi = \frac{1}{2n}.$$

Whence

$$\text{Greatest value of MN} = 2na \times \frac{1}{4n^2} = \frac{a}{2n} \text{ or } \frac{s}{4n}$$

(s being the stroke).

If $n=4$, the greatest error is only $s/16$, while if $n=12$ it is only $s/48$.

There are two cases in which there would be no error due to obliquity.

1st. With an infinite connecting rod, for then M and N would be identical. This is of course an impossibility.

2d. With connecting rod of length zero.

This latter can be in a way effected, and there is a certain practical type of direct acting engine, which we may say has a connecting road of zero length, and the movement of which is unaffected by obliquity.

The figure shows a portion of the mechanism of such an engine, used as a pumping donkey.

A is an outside view of the cylinder, with guides

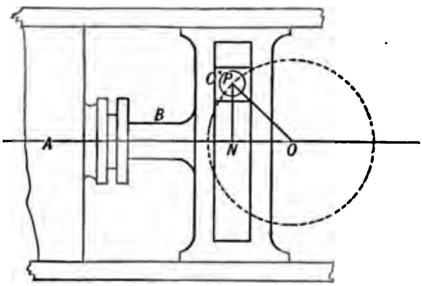


Fig. 121.

fastened to the top and bottom. B is the piston rod, the end being either forged solid or screwed into a cross-piece, thus forming a T head, in which is cut a slot at right angles to the line of stroke. C is a block sliding in the slot, which (C) in a way takes the place of the

connecting rod and contains the crank pin brasses. O is the centre of the crank shaft, the crank circle being dotted. The shaft is only shown by its centre, and the remainder of the mechanism is, for clearness, omitted.

Now, plainly this mechanism works as an ordinary direct actor neglecting obliquity.

For taking the centre point of the slot as defining the motion of the piston (page 154), this point always is at N, the foot of the perpendicular from P, the centre of the crank pin, on to the line of stroke. This point is also the centre of the end of the piston rod, so that it is C in our former figures, thus C and N are identical; and this is why we say that in a way the connecting rod is of zero length.

Having thus found that results obtained *neglecting obliquity* are not only approximate for the ordinary engine, but are also the actual results for another kind of engine, let us see what these results come to.

To find the piston position, we simply drop PN perpendicular to AA′.

But now let us proceed, as on page 160, to draw a curve.

Then (Fig. 122), we make OQ = ON, and so on. Doing this, and drawing a curve through the Q's obtained, the student will find he obtains two curves, as in Fig. 122, which look very like circles, with OA, OA′ as diameters. And in fact they are so, as can be readily proved. For

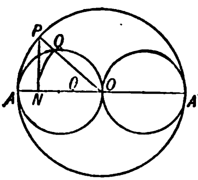


Fig. 122.

$$ON = a\cos\theta,$$
$$\therefore OQ = a\cos\theta,$$
$$= OA\cos\theta,$$

therefore AQO is a right angle.

Therefore Q lies on a circle having OA for diameter, and similarly for the other side OA′.

We see then that the determination, neglecting

obliquity, of a piston position is by far simpler than the actual accurate determination; and thus for proportions of connecting rod to crank which do not make the error very large, we can save much labour by the use of this simpler diagram.

There is one important case in which we utilise the diagram just obtained, and which we will now briefly consider.

Motion of the Slide Valve.—Fig. 123 shows the ordinary eccentric motion by which the valve is driven.

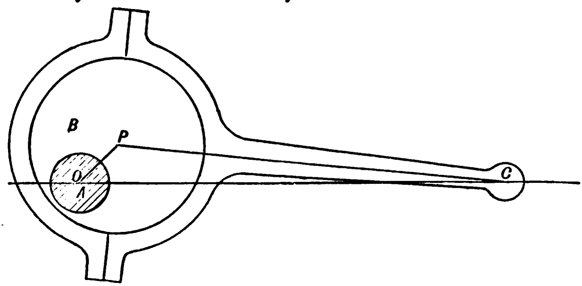


Fig. 123.

This we can easily see is identical with that of a direct actor, for

A is the shaft,
OP the crank arm,
B is the crank pin, centre P.

B is of course made very large, so that it actually embraces the whole of the crank arm in itself and projects on the other side of the shaft. But, as we have explained, the size of a bearing does not affect the motion, but only the position of its centre point. Since B however is so large it gets a new name, viz. Eccentric.

Then we have the connecting rod, now called Eccentric Rod and Strap, but being nevertheless only to us one piece, as the connecting rod was. The centre line of this is PC.

Then C is the end of the valve rod, which slides backward and forward as the piston rod does, driving the slide valve as the piston rod does the piston.

This mechanism then will be treated exactly as the direct actor ; but the proportions are such, *i.e.* the ratio CP : OP is so large, that we can, with almost absolute accuracy, neglect obliquity, and hence we shall do so.

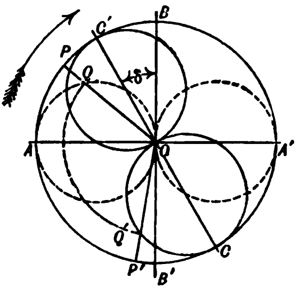


Fig. 124.

Take then (Fig. 124), AA′ = stroke of valve.

Bisect AA′ in O, and draw the two dotted circles.

These circles are the curves of valve position, and when the valve crank, or *Eccentric Radius*, as it is now called, lies along OP say, the valve will be at a distance OQ from the centre of the stroke.

Zeuner's Diagram.—But now, what we generally wish to know is the valve position for a given position of the main, not the valve crank.

To obtain this we must of course know how the two cranks lie relatively to each other. This relative position is always expressed by what is called the *Angle of Advance*, which we will represent by δ.

Then δ being given as the angle of advance, this means that the valve crank, *i.e.* eccentric radius, is, during the rotation, in advance of the main crank by an angle $\pi/2 + \delta$, so that it would in ordinary language be more correct to give $\pi/2 + \delta$ as the angle of advance ; but the technical meaning is as just explained.

In Fig. 124 then the direction of motion being as represented by the arrow, the main crank would be at OP′, where P′OP $= \pi/2 + \delta$, and the process to follow would be :—

Draw OP′, the given crank position. Measure P′OP in the direction of motion, equal to $\pi/2+\delta$, and then measure OQ.

We should then have obtained the distance of the valve from the centre of its stroke when the crank is along OP′.

[Notice OP′ is not the crank arm now in magnitude.]

But now instead of keeping the dotted curve and going through the preceding work every time we require a result, we shall save time by marking off OQ along OP′, and the same for a large number of cranks, and then drawing a curve through the points so obtained we shall have a curve whose intercepts on the main crank give the valve positions.

We will commence then to do this as follows :—

Make OQ′ = OQ.

Now, is it necessary to continue this for a large number of cranks ? The answer is No !

For what we have done is to shift Q *back* through $\pi/2+\delta$, and plainly we shall shift all the other points of the dotted curve back through $\pi/2+\delta$.

But the final result of all this will be simply to turn the two circles back through $\pi/2+\delta$, leaving them still circles.

Hence then our practical construction, viz.—

Take CC′, making $\pi/2+\delta$ with A′A as shown, then CC′ is the position to which AA′ will turn, and the two circles on OC, OC′ as diameters are the position circles.

The diagram thus obtained is called Zeuner's diagram, being due to Dr. Zeuner, and the two circles are generally called Valve Circles.

We may again repeat that its property is this. Taking any position of the main crank, the intercept between O and the circle, on the crank arm, gives the distance of the slide valve from the centre of its stroke. Its practical uses require a study of the slide valve and it appertains more properly to Engine Design.

[The student should notice that OC lies *before* OB by an angle δ. This sometimes causes a little confusion, because the eccentric being $\pi/2+\delta$ in advance of the crank, it is thought OC should be $\pi/2+\delta$ from OA′; but it must be remembered that OC comes from OA by backward turning, not from OA′.]

Piston Velocity.—After considering position, we naturally come to change of position, or motion; and rate of motion, or velocity.

We will in the present section commence with the motion neglecting obliquity, that being the simpler.

The first question will be the determination of the relation which exists between the velocity of the piston and that of the crank. We have here used the ordinary mode of expression, but strictly we should have said velocity of piston relative to cylinder and framing, and velocity of crank relative to the same; or velocity of piston-cylinder pair, and of crank-frame pair. Of course we have made the tacit assumption that the cylinder and framing are fixed. This preliminary statement being understood once for all, we shall use the common mode of statement.

In Fig. 124 we have the crank, etc. with the usual lettering.

We have first to settle how we will estimate the crank velocity.

It is an angular velocity, say A_0 per second, or we can define it by the linear velocity of the crank pin, say V_0 f.s. Then in the latter case a is the radius of reference (page 31), and

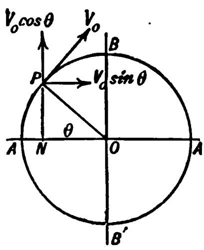


Fig. 125.

$$V_0=A_0 a.$$

Now, when at P, the *total* velocity of the crank pin is V_0 in the direction of the arrow, *i.e.* at right angles to OP.

This velocity is equivalent to a velocity $V_0 \sin\theta$ parallel to AA′, combined with a velocity $V_0 \cos\theta$ along NP.

Looking at the connection of the pin with the piston rod head, it is plain that any motion of the pin parallel to AA′ must cause an equal motion of the head in that direction. Hence

Velocity of piston rod parallel to $AA' = V_o \sin \theta$.

But the piston rod can only move parallel to AA′, so that this velocity is its *total* velocity; and hence calling the piston velocity at the instant considered V_p we have

$$V_p = V_o \sin \theta.$$

[With regard to $V_o \cos \theta$, the other part of V_o. This part does not cause any motion of the rod, for the pin is free to slide along the slot in the direction NP. It follows then that the pin having a velocity $V_o \cos \theta$ in that direction, and the piston rod head having necessarily no velocity in that direction, $V_o \cos \theta$ is the velocity with which the pin, and necessarily the block encircling it, slides along the slot.

So sliding velocity of block-piston rod pair $= V_o \cos \theta$.

We should here particularly notice that V_o is the *total* velocity of the pin, hence we can resolve it into its components as we have done. But suppose we had looked at the question in another way as follows :—

The piston moves forward with a velocity V' say, therefore resolving V' in the direction of motion of P we obtain,

$$V_o = V' \sin \theta.$$

Then we should be wrong, the reason being that V' represents only a part of the motion causing the motion of the pin. In addition to moving along with V', the pin must necessarily slide along the slot with some velocity V'', and then V_o is the combination of the two, thus V_o contains also a term representing the effect of V'' in the direction of V_o, and the total result is

$$V_o = V' \sin \theta + V'' \cos \theta.$$

This agrees with what we before obtained. For

$$V' = V_o \sin \theta, \quad V'' = V_o \cos \theta,$$
$$\therefore V' \sin \theta + V'' \cos \theta = V_o \sin^2 \theta + V_o \cos^2 \theta = V_o,$$

so that we obtain the same result.

The foregoing is a very common error, and should be carefully guarded against.]

Since $V_p = V_o \sin \theta$, we have

$$\text{Velocity ratio of piston to crank} = \frac{V_o \sin \theta}{V_o},$$
$$= \sin \theta \text{ (radius of reference } a).$$

We want now a graphical representation of this, and there are two ways in which we may look at it.

1st. **Let V_0 be constant.**—Then we can represent this constant velocity by a line of given length ; generally we take OP to represent V_0 and then the scale is

$$\text{OP in inches to } V_o \text{ f.s.},$$

or

$$1 \text{ inch to } V_o/\text{OP f.s.},$$

OP here standing for the number of inches it measures in the diagram, and having nothing whatever to do with that length—the crank arm—which on another scale it represents.

Thus if $a=2$ ft., and we draw OP=2 ins., then the linear scale is 1 in. to 1 ft. Now if $V_o=20$ f.s. say, the velocity scale would be 2 ins. to 20 f.s., not 2 ft. to 20 f.s.

OP then representing V_0, if we draw PR parallel to AA′, cutting OB in R, we obtain

$$\text{OR} = \text{OP} \sin \theta,$$

therefore OR represents the piston velocity. We proceed then to draw a curve of piston velocity by making, on OP, OQ equal to OR, and so on for a large number of cranks as shown in Fig. 126.

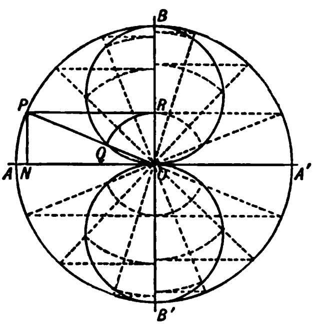


Fig. 126.

Then a curve drawn through the Q's is a curve of piston velocity, the intercept which it cuts off on any

particular crank giving the piston velocity corresponding to that particular crank position.

It is simple to prove, as in the position curve, that the curve consists of two circles, but on OB and OB′ now as diameters, instead of OA and OA′.

The piston velocity V_p then varies from

Zero at A and A′

to

V_o at B and B′,

and is found for any intermediate crank position OP by measuring OQ on the particular velocity scale previously described.

The inverse proposition, to find the crank position for a given piston velocity, is equally simple. For take OQ on the velocity scale to represent the given velocity, and then describe a circle with centre O, radius OQ, which will cut the curve in four points, either of which joined to O gives a crank position fulfilling the required condition.

The curve just drawn gives us crank positions, and if piston positions be either data or required results, we should have first to find the crank position corresponding to the given datum, and then deduce the velocity or piston position required.

This process is simple enough, but we can render it even simpler by drawing a curve, once for all, which shall give velocities corresponding to given piston positions, and *vice versa.*

To do this we must proceed thus :—

Taking any piston position N (Fig. 126), set up NP perpendicular to AA′, and along NP mark off the piston velocity corresponding to the piston position N. But since OP is the corresponding crank, this piston velocity is $V_o \sin \theta$, *i.e.* NP itself; and thus the curve drawn through the points so obtained is the crank circle itself.

We have then two velocity curves :—

One in which intercepts are measured from a point O, or a pole; hence called the **Polar** curve.

And one in which intercepts are measured from a base line; hence called a **Linear** curve.

2d. **Let V_0 be not constant.**—We cannot now represent V_0 by a line of constant length. But the work by which we obtained the velocity *ratio* did not depend on the constancy or otherwise of V_0, and hence we still have

$$\text{Velocity ratio of piston to crank} = \sin\theta,$$

and if we take such a scale that OP represents unity, OR (Fig. 126) will represent $\sin\theta$.

We can then still draw the curve as usual, but OQ now does not give the velocity of the piston, but only the ratio of its velocity to that of the crank on the scale just explained.

The preceding is not of very great practical importance, since V_0 is in most cases so nearly uniform that we may without sensible error assume it to be exactly so.

Mean Piston Velocity.—Referring to chap. i. (page 22) we have for this simply

$$V_m = \frac{\text{stroke}}{\text{time of stroke}}.$$

[Stroke meaning either the backward or forward movement, but not the whole.]

But

$$\text{time of stroke} = \text{time of half revolution},$$

$$= \frac{\pi a}{V_0} \quad (\text{since } \pi a \text{ is the length of the path of pin}),$$

$$\therefore V_m = \frac{2a}{\frac{\pi a}{V_0}} = \frac{2}{\pi} V_0.$$

But V_0, besides being the crank speed, is also the maximum piston speed,

$$\therefore \text{Ratio of maximum to mean piston speed} = \pi : 2.$$

Actual Motion with Obliquity.—We must now see what results we obtain in the actual motion, taking account of the obliquity, and we can also see how they compare with our preceding simple ones.

To obtain the relation between V_p and V_o, we cannot proceed quite so simply as we did before.

Drawing the usual figure, we have C moving towards O with velocity V_p, and P perpendicular to OP with velocity V_o.

Now there is a connection between V_p and V_o, because P and C are connected together, and the relation which exists between them depends on the particular

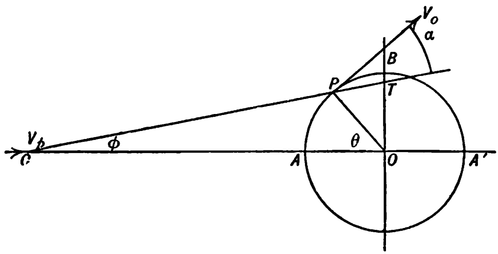


Fig. 127.

method of connection. We must then examine into the connection, and when we do so, we see that its essential characteristic is that C and P cannot approach or recede from each other, the distance CP being constant.

From the preceding it follows that if, at any instant, C have a velocity in the direction CP, P must have the same velocity in that direction. Because if it were not so, then P would be either approaching to or receding from C; and this is not affected by either C or P, or both, having in addition velocities at right angles to CP, because no finite velocity at right angles to CP can be at the instant altering the length of CP. There may be a little difficulty experienced here, one cause perhaps

being that we speak of velocity along CP, while CP is a rod which is continually changing its direction; this difficulty can be removed if we are careful to remember that the whole question relates to an *instantaneous* state of affairs, and that the direction CP does not mean the solid rod, because of course neither C nor P can move along the rod, but it means the line along which the rod at the instant lies, and it is a perfectly definite direction, in spite of the fact that, at the next instant, the rod will be along some other line. Some aid to understanding the point can perhaps be got from splitting the instantaneous motion of the rod into two parts; the first a bodily translation along CP, in which plainly C and P have the same velocity; and the second a turning of the rod, so as to keep C up to the line of stroke, and P down on the crank circle, this latter consisting only of motions of C and P at right angles to CP.

We have then

The total velocity of C is V_p along CO, which resolves into

$$V_p \cos\phi \text{ along CP,}$$

and

$$V_p \sin\phi \text{ perpendicular to CP.}$$

The total velocity of P is V_o at right angles to OP, which resolves into

$$V_o \cos\alpha \text{ along CP produced,}$$
$$V_o \sin\alpha \text{ perpendicular to CP} \quad \text{(Fig. 127).}$$

We have seen that the two first components must be equal,

$$\therefore V_p \cos\phi = V_o \cos\alpha.$$

Let now CP produced cut OB in T. Then

$$\begin{aligned} \cos\phi &= \sin \text{OTP}, \\ \cos\alpha &= \sin \text{TPO}, \\ \therefore V_p : V_o &= \sin \text{TPO} : \sin \text{OTP}, \\ &= \text{OT} : \text{OP}, \end{aligned}$$

which gives us the velocity ratio, and if V_0 be constant and we take OP as before to represent it, then OT represents V_p.

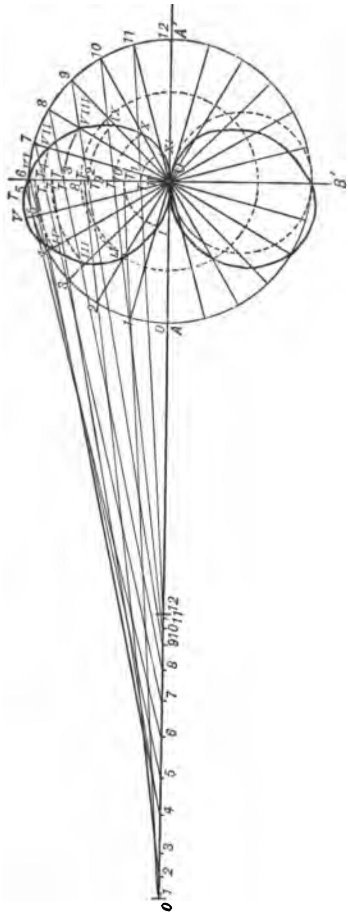


Fig. 128.

We can now proceed as in the preceding to draw a curve of velocity, or of velocity ratio, by taking OQ along OP, equal to OT, and doing this for a large number of cranks, draw a curve finally through all the Q's. The detailed construction is shown in Fig. 128. We explained very fully in the preceding case, and so we need not again do so. The crank circle is divided in 1, 2, etc.; then 11, 22, etc. are the connecting rods, which being produced cut OB in T_1, T_2, etc. Then we make $OI = OT_1$, $OII = OT_2$, etc., and finally draw the curve through I, II, etc., the lower half being symmetrical with the upper. The

curve so obtained is, if V_0 be constant, a curve of piston velocity, scale $OP = V_0$.

If V_0 be not constant, it is a curve of velocity ratio of piston to crank, scale OP = unity (compare page 174).

We can now see what effect obliquity has, by comparing this curve with the one obtained, neglecting obliquity. To do this draw the dotted circle on OB′. We then see that so long as OP is to the left of OB, the curve lies outside the circle, so the velocity is greater than that obtained neglecting obliquity. Also the maximum velocity is greater than V_0, and occurs at a crank position before OB. When OP is along OB, $V = V_0$ as it did before. When OP is to the right of OB, then the velocities are all less than they were when we neglected obliquity.

The mean velocity is of course not affected by obliquity, and is always

$$V_m = \frac{2}{\pi} V_0,$$

and is represented by the dotted circle.

We can now obtain, if necessary, a linear diagram of velocity, on the stroke as base, referring to piston position.

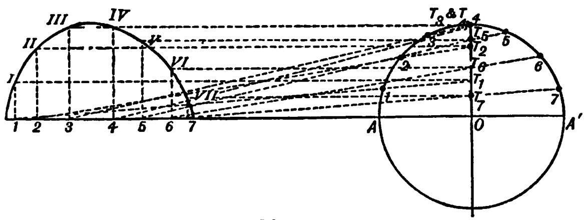


Fig. 129.

Fig. 129 shows the construction, and any further explanation can be got by referring to page 170, where the diagram is explained for the case neglecting obliquity.

Angular Velocity of Connecting Rod.—The velocity here mentioned is sometimes required, and the

process of finding it will enable us to introduce an important method, so we will briefly consider it.

A pin in the end of the rod turns in a bearing in the crosshead, or *vice versa*, and the angular velocity we wish to obtain is that of the pin relative to the bearing. Now CP is the rod, and V_p, V_o the velocities of its ends. The crosshead bearing has also the velocity V_p, and this causes our difficulty, because we have a moving bearing. Now the method we use depends on the following principle: We cannot alter the relative motion of two bodies by imparting to them as a whole a

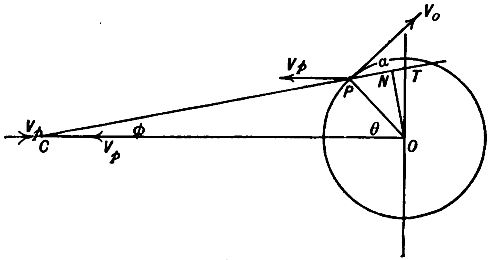


Fig. 130.

velocity in any given direction. This simply means that supposing the two bodies mounted say, on a stand, then we cannot by moving the stand about affect the relative motion of the bodies.

Apply this principle then, by imparting to the connecting rod and crosshead as a whole a velocity V_p parallel to OC.

Then this leaves C with resultant velocity zero, so the bearing is reduced to rest, while P has now a velocity composed of V_p and V_o. To find the resultant velocity of P resolve V_p and V_o along and at right angles to CP. Then along CP

$$\text{Resultant velocity} = V_o \cos \alpha - V_p \cos \phi,$$
$$= 0 \quad \text{(page 173)},$$

which of course it should be, since, C being at rest, P can only move at right angles to CP. This verifies our principle. We have then

$$\text{Velocity of P is at right angles to CP} = V_o \sin \alpha + V_p \,.\, \sin \phi,$$

and graphically

$$= \text{OP} \sin \alpha + \text{OT} \sin \phi,$$
$$= \text{PN} + \text{NT}$$
(ON being perp. to PT),
$$= \text{PT} \,.\, \text{on the velocity scale.}$$

So the third side PT of the triangle OPT represents the linear velocity of P relative to the crosshead.

We can at once deduce the angular velocity of the rod or pin in the bearing.

For, calling it A,

$$A = \frac{\text{Velocity of P}}{\text{CP}},$$

because the end C is now still,

$$\therefore\ A = \frac{\frac{\text{PT}}{\text{OP}} \,.\, V_o}{\text{CP}} = \frac{\text{PT}}{\text{CP}} \,.\, A_o.$$

So the velocity ratio of connecting rod-crosshead pair to crank-frame pair is PT : CP.

One result which we can at once obtain is the speed of rubbing of the gudgeon pin in the bearing at the instant. For if r be the radius of the bearing in feet,

$$\text{Speed of rubbing} = Ar = \frac{\text{PT}}{\text{CP}} \,.\, A_o r \text{ f.s.}$$

We may if necessary construct curves showing this angular velocity, as has already been done for piston velocity.

EXAMPLES.

(All to be solved by construction when possible.)

1. The stroke of an engine is 4 ft. 3 ins., length of connecting rod 8 ft. The steam is to be cut off at .6 of the stroke. Find the angle the crank then makes with the line of stroke. If the

crank position be found neglecting obliquity, and the cut-off take place at the position so found, what will be the actual point of cut-off? *Ans.* 86° 13′; .665.

2. Show that the greatest error in piston position due to obliquity is very approximately $\frac{s}{4n}$; s being the stroke, and n the ratio of connecting rod to crank. Apply this to the preceding example. *Ans.* $3\frac{3}{4}$ ins.

3. In (1) the travel of the slide valve is 10 inches, and the angle of advance 20°. Find the piston position when the valve is in the middle of its stroke, and also when it has moved $2\frac{1}{2}$ inches either way from the central position.

Ans. 19.8; .41; 43.9 ins. from beginning of stroke.

4. The stroke of a pumping donkey is 8 inches, and it runs at 85 revolutions per minute. Find the speed of the piston at each eighth of its stroke.

Ans. 117.7, 154.2, 172.3, etc., ft. per min.

5. Draw to scale a curve of velocity for an engine 3′ 6″ stroke, 7′ 6″ connecting rod, running at 110 revolutions. Draw the crank circle on a scale of 1 in. to 1 ft., and state what scale this causes the curve to be—1st, as a velocity ratio curve; 2d, as a velocity curve. Find the maximum piston velocity, and the velocity at each quarter stroke. Also find the piston position when the actual velocity equals the mean.

Ans. Scales, 1st, $1\frac{3}{4}$ ins. to unity; 2d, 1 inch to 691.4 ft. per min. Maximum velocity, 1332; 1st quarter, 1141; 2d, 1320; 3d, 968 ft. per min. Piston positions, 4, $35\frac{3}{4}$ ins. from beginning of stroke.

6. Construct a curve of angular velocity of connecting rod by marking off the values of PT (page 176) along the corresponding cranks for the preceding. State on what scale this curve gives, 1st, angular velocity of rod; 2d, speed of rubbing of gudgeon, diameter, 10 ins. Give numerical value of the maximum in each case.

Ans. Scales, 1 inch to 1.536 radians per second; 1 inch to .64 f.s. Maximum values, 2.69 radians per sec.; 1.12 f.s.

7. Show, in a pumping donkey, that the curve giving the speed of rubbing of the crank pin brasses, in the slot in the piston rod head, is a pair of circles.

8. If A_o be the angular velocity of the crank and A that of the connecting rod at a given instant; show that the angular velocity with which the crank pin is then rotating in its bear-

ing is $A_o - A$; regard being had to the sign of A, taking the direction of A_o as positive.

Ans. Applying a velocity V_o to the whole engine at right angles to OP reduces the centre of the bearing to rest. O moves with velocity V_o, so the crank pin rotates round P with angular velocity A_o; and C, having a velocity compounded of V_p and V_o, it will be found is rotating round P with angular velocity A (page 177). Since pin and bearing both rotate round the same centre, the relative velocity of rotation is evidently the difference of A_o and A if they are in the same direction, and the sum if in opposite directions. The relative velocity is then $A_o - A$.

This is one case of a very important principle, viz. that the relative angular velocity of two bodies as above is the difference of their angular velocities, *irrespective of what their centres of motion may be.* The general principle can be proved in the above manner.

9. Find the velocity of rubbing of the crank pin of No. 5 in its bearing when on either dead centre. Diameter 14½ ins.

Ans. 8.6; 5.33 f.s.

CHAPTER IX

THE DIRECT ACTOR, CONTINUED—FORCES—CRANK EFFORTS

HAVING investigated the relative motions of, we next proceed to consider the forces which act between, the pieces of the engine.

The effort driving the engine is the total steam pressure on the piston. The resistance is a moment applied in some way to the crank shaft; it may be due to an actual moment, as the resistance of the water to the turning of a screw propeller; or it may be caused by some linear resistance, such as the resistance to a cutting tool of a planing machine, this being transmitted back along the mechanism of the machine to the shop shaft, becoming a resisting moment. In all cases, no matter what the original source be, the resistance finally shows itself as a moment applied to the shaft.

We proceed to investigate the action under Balanced Forces and neglecting friction. Let

M = resisting moment. P = total piston pressure.

Then we have

Energy exerted = work done.

Now, as we have before stated (page 137), we must always define clearly what period of time we are going to consider. This at once leads to the question, Are P and M constant or variable? For without this knowledge we cannot tell for any period what the energy or

work will be. Actually we know that P is rarely, if ever, constant; and, as we shall soon see, if P were so, M could not be. In all investigations, however, we commence with simple cases, and so we will suppose that P is a constant pressure during the whole stroke.

Again, for balanced forces, there must either be no change at any instant in the motion of the parts of the machine, or else we must select such a period of time that there is no change on the whole (compare page 96). If we kept strictly to this we should be only able to consider a whole revolution in which to compare energy

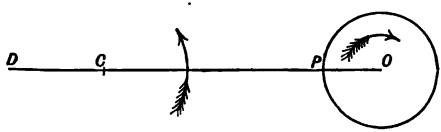


Fig. 131.

and work. For at the commencement of the stroke, drawing Fig. 131 with the usual lettering—

OP is revolving clockwise round O,
PC ,, anti-clockwise round C,

and CD (the piston and rod) is still. Now CD begins moving, and is not still again till the end of the forward stroke, and then OP is revolving clockwise as before, but CP is now also revolving clockwise (Fig. 132). So

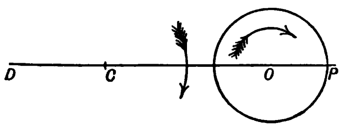


Fig. 132.

there has been an effect produced on the mechanism (page 60); and we cannot reach a stage of no effect produced until we reach the position of Fig. 131 again.

We are then debarred from comparing together any-

thing more than the mean values of P and M during a whole revolution. For this we have, P being constant,

$$\text{Energy exerted} = \text{P} \times 2s \text{ or } \text{P} \times 4a, \quad (s = \text{stroke}, \ a = \text{crank arm as usual}),$$

$$\text{Work done} = \text{M}_m \times 2\pi,$$

M_m being the mean value of M,

$$\therefore \ 2\pi \text{M}_m = 2\text{P}s.$$

But we want if possible to examine more closely than this, even if we can only obtain approximate results, and this we do by agreeing for the present to consider the pieces as weightless. Then if they have no weight it does not matter whether their motion be changed or no,

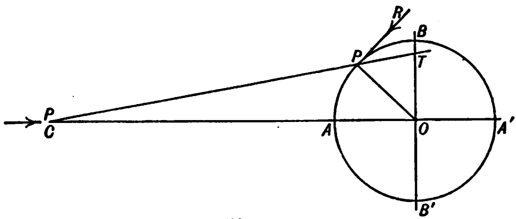


Fig. 133.

because no energy can be expended on them. We say then that we neglect the weight of the parts.

We are now at liberty to apply the principle of Balanced Forces to any period we please, and the period we will select is the indefinitely short one, while the engine is in the position of Fig. 133.

Since the effect of a moment is independent of the forces of which it consists (page 65), we are at liberty to suppose the moment M applied to the shaft by means of a force R at right angles to OP, acting at P, and such that

$$\text{R}a = \text{M}.$$

Then R exactly replaces the moment, so far as resistance to turning is concerned.

The Principle of Work, we have seen, may be written

$$\text{Force ratio} = \frac{1}{\text{velocity ratio}} \quad \text{(page 96).}$$

Whence for the small period considered, *i.e.* at the instant,

$$\frac{R}{P} = \frac{\text{velocity of piston}}{\text{velocity of point where R is applied}}.$$

But the latter is the velocity ratio of piston to crank, and equals OT/OP (page 173). Hence

$$R : P = OT : OP.$$

[And we may notice that this holds irrespective of the constancy or otherwise of P.]

Crank Effort.—Since the forces are balanced it follows that the *useful* effect of P must be to produce, at the end of OP, a force exactly equal and opposite to R. This force is called the **Crank Effort.**

It must be particularly noticed that we say the *useful* effect of P; this is because the turning of the arm against R is the useful effect required. The crank effort is not the total effect of P on the shaft, as we shall see a page or two farther on.

The numerical value R then, found above, represents either the resistance, or the crank effort, as we take it in one or the other direction.

Looking back to page 173 we found there

$$V_p : V_o = OT : OP,$$

and we proceeded to draw a curve of piston velocity, OP representing the constant V_o.

Plainly then, since we now have

$$R : P = OT : OP,$$

if we take OP to represent the constant P, OT represents R; and the curve of piston velocity becomes a

curve of crank effort. Having then once constructed this curve, by the rules given in the last chapter, it may be used either to find piston velocities or crank efforts, using of course in each case the proper scale, *i.e.*

For velocity, OP ins. $=V_o$ f.s. For force, OP ins. $=$ P lbs.

Before proceeding farther we will obtain the preceding result by another method, not so concise as that

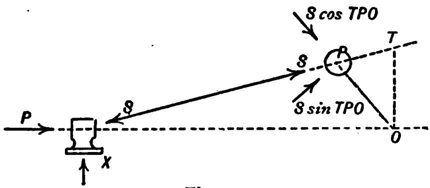


Fig. 134.

just used, but not requiring a previous knowledge of the velocity ratio.

In Fig. 134 consider the equilibrium of the crosshead (recollecting it is weightless).

The pieces touching it are (see page 97),

	Connecting Rod,	Piston Rod,	Guide.
Forces	S,	P,	X.

For a horizontal balance

$$S \cos \phi = P,$$
$$\therefore \ S = P \sec \phi.$$

Next, we have defined crank effort as the useful effect produced by P on the crank arm. Now the first effect of P is to produce a thrust S on the connecting rod end, which, together with the equal and opposite S in the figure, constitutes the action or stress between the bearing surfaces of gudgeon pin and bearing (page 106). This stress S is transmitted along the rod, and finally produces a stress S between connecting rod brasses and crank pin.

The crank pin then is acted on by a force S.

Resolve this S into components—

$$S \sin TPO \text{ at right angles to } OP,$$
$$S \cos TPO \text{ along } PO.$$

Then the first is the only one which produces any useful turning effect, it is therefore the crank effort, and

$$\begin{aligned}\text{Crank Effort} &= S \sin TPO,\\ &= P \cdot \frac{\sin TPO}{\cos \phi} = P \frac{\sin TPO}{\sin OTP},\\ &= P \cdot \frac{OT}{OP}.\end{aligned}$$

This result verifies then, in this case, the Principle of Work, and also by this method of proceeding we have traced the effect of P, showing how it first acts to cause a stress between crosshead and gudgeon pin, and then this stress is transmitted along the rod, finally causing the crank effort.

Mean Crank Effort.—Representing the mean crank effort by R_m we have

$$\begin{aligned}R_m \times a &= M_m \text{ (the mean moment)},\\ &= \frac{4Pa}{2\pi} \text{ (page 182)},\\ \therefore R_m &= \frac{2}{\pi}P.\end{aligned}$$

It is usual to complete the curve of crank effort by drawing a circle to represent R_m. The radius will be $2 . OP/\pi$, and thus the dotted circle of mean velocity (Fig. 128) is also the circle of mean crank effort.

Linear Curve of Crank Effort.—After obtaining the polar curve of velocity ratio, we (page 175) drew a linear curve on the piston stroke as base, and we could similarly draw a curve of crank effort. But such a curve would be of little use, and we will draw a linear curve in a different manner.

The crank effort exerts energy on the crank pin; and, although it moves in a circular path, yet its direc-

tion is always along the path. Referring then to chap. iii., page 71, if we straighten out the path of the pin, and set up at every point an ordinate representing the effort, when the pin is at that point; we shall obtain a curve which will give us, not only the crank effort for any position of the pin, but also the energy exerted by the effort between any given positions. This latter will be represented as shown in chap. iii. by the area between the two given ordinates.

In Fig. 135 we have drawn a polar curve of crank effort, OA representing P (the piston pressure).

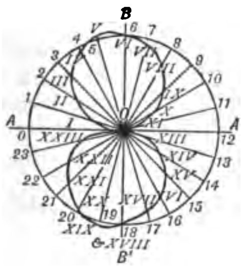


Fig. 135.

Now, in Fig. 136, lay off the crank circle unrolled as a base. The points ABA′B′ being named identically in the two figures.

Divide the circle in 135 into a number of equal parts, say 24; and also mark off the same parts on the unrolled circle in 136. Now measure in 135 the efforts, and set them up as ordinates at the corre-

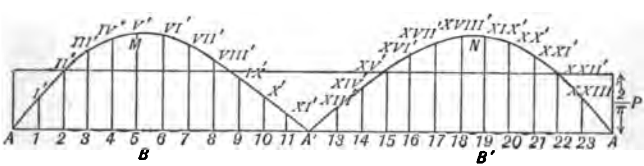


Fig. 136.

sponding points of 136 as shown; 1I′, 2II″, etc., being identical with OI, OII, etc.

Finally, in 136, draw the curve AI′II′MA′NA through the tops of the ordinates.

The curve AMA′NA is now a linear curve of crank

effort. But it gives the efforts at points of the path of the crank pin, not of the piston.

We complete by drawing a line, parallel to the base, at a distance $2P/\pi$ or $2OA/\pi$ from it, showing the value of the mean crank effort.

The full curves drawn are the true curves, taking account of obliquity, the actual case they refer to being $n=4$ (n being connecting rod—crank ratio). If we neglect obliquity the polar curves, as we have seen, are circles (Fig. 128); and the derived linear curve is obtained as above.

Looking now at the shape of the curve, we see that, even with a constant steam pressure, the driving effort is very irregular. Thus, by reference to Fig. 128, it will be found to vary from 0 at A and A′ to a maximum equal to P at B and B′, *i.e.* from 0 to $\pi/2$ times the mean value—when obliquity is neglected; and from 0 to 1.57 times the mean, when obliquity is allowed for (Fig. 135). With a shorter rod the effect would be still worse, but the ratio 4 to 1 is about the smallest ever practically used. The result of altering the length of the rod is shown in the table on page 195.

Motion of the Crank Shaft.—If the crank shaft revolved with exact uniformity, it would be necessary that, at every instant, the moment R*a* of the crank effort should be balanced by an exactly equal resisting moment; and in the first method of finding the ratio of R to P (page 183) we assumed this to be the case. But the assumption there made was necessary, only that we might take the whole engine as the body acted on—or at least the whole of the moving parts, *i.e.* piston and rod, connecting rod, and crank shaft.

But now applying the first method to the body consisting of piston and rod, and connecting rod only; we shall find the same value for R as before, quite irrespective of what the resisting moment applied to the shaft is. Or looking at the second method (page 184) of

obtaining R, that is, as it stands, independent of the resisting moment.

Now, in actual practice, the resisting moment does not vary at all according to the crank effort, but depends on entirely different considerations; and, as a rule, the moment is very nearly uniform. It follows then that, the driving effort being irregular and the resistance uniform, there must be produced an irregularity of motion in the shaft. What the magnitude of this effect will be we have so far not the means of calculating; but we can see that the effect will be an irregularity. For example—at A and A′ there is no effort at all, and hence if means be not provided for moving the crank over these points, the engine will stop; hence the name *dead points*. The irregular application of effort also causes a greater tendency to break, in the parts, than a regular application would.

Two-Crank Engine.—One way in which the irregularity of action can be decreased is by using two cylinders, working on cranks at right angles, on the same shaft. Then, for example, when one crank is on the dead point the other will have almost its maximum turning effect.

We will now see how to represent the combined turning effect of the two cylinders.

In Fig. 137 we have the two crank efforts R_1 and R_2. These produce turning moments R_1a, R_2a, both clockwise. Hence, no matter what the relative directions of the R's are, their combined effect is a clockwise turning moment, $R_1a + R_2a$.

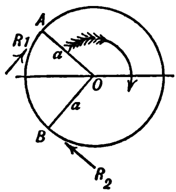


Fig. 137.

$$\therefore \text{Combined moment} = (R_1 + R_2)a.$$

Now this moment we can represent graphically by assuming it to consist of a crank effort $R_1 + R_2$, acting either on OA

or OB, because it is actually equivalent to such an effort.

We can now draw a curve by setting off the value of the combined crank effort along either OA, the *leading*, or OB, the *following* crank. It is quite immaterial which we choose, but we must keep to the same one all through the construction.

The process then will be as follows :—

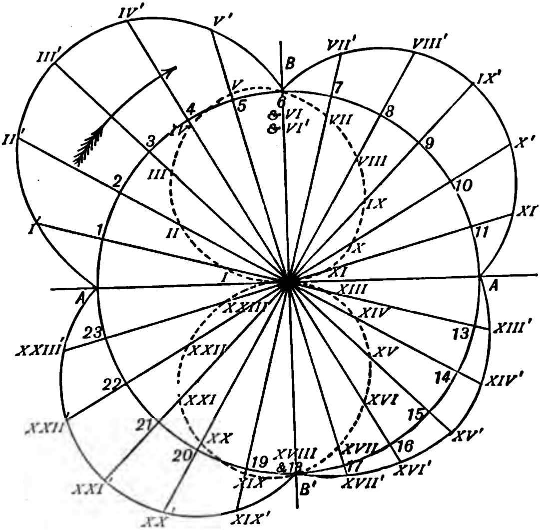


Fig. 138.

In Fig. 138 we have drawn the ordinary single diagram of crank effort in dotted lines. And we will choose the leading crank as that along which to set off the combined effort.

Commence with the leader at OA. The follower is at OB′.

$$R_1 \text{ (leader)} = 0, \quad R_2 \text{ (follower)} = OB',$$
$$\therefore R_1 + R_2 = OB', \quad \text{and set off } OA = OB'.$$

A is the first point on the combined curve.

Next—Leader at O1, Follower at O.19.

$$R_1 = O\,I, \quad R_2 = O\,XIX.$$
$$\therefore R_1 + R_2 = O\,I + O\,XIX, \text{ and set off } II' = O\,XIX.$$

Then $OI' = R_1 + R_2$, and I′ is the next point on the curve.

So on for II′, III′, etc.

Finally draw the full curve through A, I′. . . .

We have now drawn a curve such that the in-

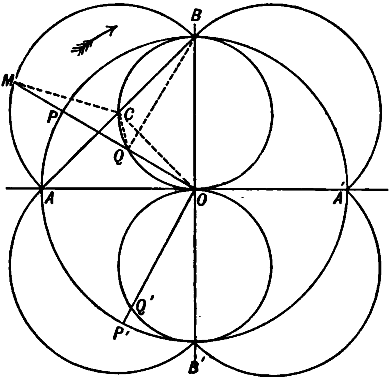


Fig. 139.

tercept cut off by it on any radius gives the combined crank effort, when the *leading* crank lies along that radius. The actual example is for $n = 4$, and we will next see what the curve becomes, neglecting obliquity.

This being a simple case, we will, for a change, solve it geometrically.

Drawing the two circles, which constitute the single diagram for this case, join AB, cutting the circle in C, and join OC.

Then evidently OC bisects AB at right angles, and the angle AOC is $\pi/4$.

Let now OP represent the leading crank, then OP′ (the follower) is $\pi/2$ behind it.

$$\therefore\ \angle POP' = \frac{\pi}{2},$$

therefore BQ and OQ′ are parallel,

$$\therefore\ BQ = OQ'.$$

Now produce OQ, and make QM = OQ′ or BQ. Then OM is the combined crank effort.

Join CQ and CM. Then $\angle CQB = \angle COB$ (in same segment).

$$\therefore\ CQB = \frac{\pi}{4},$$

and

$$BQM = \frac{\pi}{2}, \quad \therefore\ CQM = \frac{\pi}{4}.$$

Therefore in the triangles CQB, CQM,

$$\begin{gathered} QM = QB. \\ QM \text{ is common,} \\ \angle CQM = \angle CQB, \\ \therefore\ CM = CB, \end{gathered}$$

therefore CM, CB, CA, and CO are all equal.

This proves that M lies on a circle with centre C and radius CO, CA, or CB. Hence this circle is, for the first quadrant, the curve of combined crank effort; and from symmetry the four parts of the curve will be similar to each other.

We have then the full curve, consisting of four arcs, as shown in Fig. 139.

Mean Combined Effort.—Since there are now two cranks, the mean combined effort will be simply twice that for one crank.

$$\therefore\ R_m = 2 \times \frac{2}{\pi}\,P = \frac{4}{\pi}.\,P.$$

Or,

$$\text{Energy exerted} = \text{P}s + \text{P}s \quad \text{(in one stroke)},$$
$$\therefore\ \text{R}_m = \frac{2\text{P}s}{\pi a} = \frac{4}{\pi}\ \text{P}.$$

We will now combine our results by placing on one figure (Fig. 140(a)), the curves, taking account of obliquity (in full), and neglecting obliquity (dotted); and draw the dotted circle of mean crank effort. In order that there should be no doubt as to which crank the effort is laid off along, we mark in each quadrant the position occupied by the cranks at the centre of the quadrant. It is then in our case clear that the efforts are for

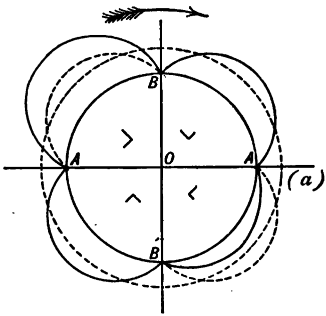


Fig. 140 (a).

positions of the leading crank, since in each case it is that crank which lies in the quadrant, the follower lying across it.

Examining the diagram we see that in both the full and the dotted figures there are four minimum values at the quarter revolutions, each equal to P; and four maxima at the eighths. In the dotted curve the four maxima are equal; but in the full, calling the quadrants 1st, 2d, etc., from OA, the greatest maximum is in the 1st, and the least maximum in the 3d; while those in the 2d and 4th are the same with as without obliquity,

the dotted and full curves being practically coincident, as the figure shows.

There are eight positions in which the actual effort coincides with the mean; and the deviation from the mean is very much less than when only one crank is used. The variation is now from a minimum $\pi/4$ times the mean to a maximum 1.31 times the mean.

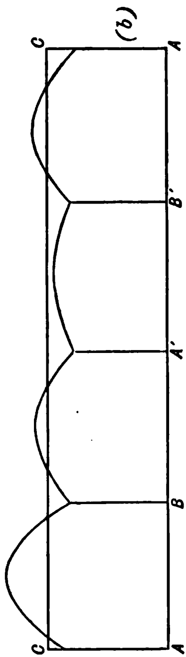


Fig. 140 (*b*).

Linear Curve.—By exactly the same method as that used for the single crank, we can draw the linear curve for the two cranks. The base is the unrolled crank circle, and it represents now the path of the *leading* crank pin. The radii are transferred exactly as before described, and no further description is now required.

We thus obtain Fig. 140 (*b*), where we have taken only the actual case, not that neglecting obliquity, the latter having been sufficiently considered already.

The line of mean effort is now at a height $4P/\pi$.

The construction of the linear curve could have been proceeded with, if we had pleased, directly from the single linear curve (Fig. 136) by addition of the ordinates; we could thus avoid drawing the combined polar curve, and the linear being the one, as we shall see, of most use, this method is often followed.

By following an exactly similar process to the preceding, we could represent the effect of three cylinders working on a single shaft. The cranks in such a case

may be arranged in various relative positions, but the one now practically universal is to have the cranks lying at 120° apart. In this case (Fig. 141) it is evidently impossible to call either the leading crank, and so they must be distinguished in some other way. Usually they will belong to a triple expansion engine, and we distinguish them as high, intermediate, and low.

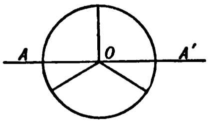


Fig. 141.

It will be noticed, however, that in construction it is unnecessary to specify along which crank the combined effort is set off. For looking back to Fig. 138, when one is at OA, one is at O8, and one at O16, we add these three and set them off along OA say. Then we add O1, O9, O17, and set off along O1, and so on; till O7, O15, O23 are set off along O7. Now we come again to O8, O23, OA to set off now along O8, and so we get the same sets of three over again starting from O8 instead of OA. And we shall come to the same again at O16. So whether we start by setting off along OA, or O8, or O16, exactly the same curve will result.

The actual construction of this case we will set as an example, and it will be found that the ratio of maximum to mean crank effort is still further reduced. We append in tabular form results showing the effect of number of cylinders, and of connecting rod length, on the regularity of the turning effort. We assume equal pressures in the cylinders, and that the pressure is uniform throughout the stroke. For the method of dealing with the question, when the pressures are unequal and varying, we must refer to the larger treatise. It may be noticed, however, that equality of total pressure in the two cylinders of a compound, or the three of a triple expansion engine, is one of the conditions the designer aims at. So the assumption of such an equality is not at variance with the practical facts.

Fluctuation of Crank Effort with Uniform Steam Pressure.				
Ratio to Mean for	One Crank.	Two Cranks at Right Angles.	Three Cranks at 120°.	Length of Connecting Rod.
Maximum Minimum	1.57 0	1.112 .785	1.047 .907	Infinite
Maximum Minimum	1.62 0	1.31 .785	1.077 .794	Four Cranks
Maximum Minimum	1.59 0	1.24 .785	1.05 .81	Six Cranks

Examples.

1. The curve of question 5, p. 178, is also a curve of crank effort. If the diameter of the cylinder be 68 inches, and the constant effective steam pressure 18 lbs. per square inch, find the scale of the curve; and construct carefully combined diagrams, both polar and linear—1st, for two such cylinders on cranks at right angles; 2d, for three cylinders on cranks at 120°.

The cylinders are not of the same diameter, but the total pressure is taken to be the same (page 194).

Give numerical results for the maximum, minimum, and mean crank efforts in each case.

Ans. Scale 1 inch to 16.68 tons—

$$\left.\begin{array}{llll} 1 \text{ Crank,} & 30.23, & 0, & 18.6 \\ 2 \quad ,, & 48.6, & 29.2, & 37.2 \\ 3 \quad ,, & 59.8, & 43.1, & 55.8 \end{array}\right\} \text{tons.}$$

2. Draw a combined curve for the second case of the above when two of the cranks are at right angles, and the third makes equal angles with the other two. Compare the regularity of the effort in the two cases.

Ans. Maximum to mean 1.28 as against 1.07.
Minimum to mean .6 as against .77.

The value of the graphic method depends not only on the facility with which single results can be obtained, but also on the obtaining and preservation of a continuous set of results. Hence the curves above should be carefully and accurately drawn, and inked in for future reference.

CHAPTER X

UNBALANCED FORCES

IN the present chapter we shall investigate the manner of dealing with cases in which the energy applied to a body or machine does not leave it in the same state, at the end of the time considered, as it found it in at the commencement—or in other words, where the forces are not balanced.

Taking the simplest case, Fig. 142, A is a sliding piece, moved by an effort P, resisted by a resistance R. P and R not being equal, what kind of motion will ensue?

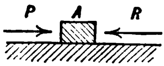


Fig. 142.

Take a movement x of the slider. Then,

$$\text{Energy exerted} = Px.$$
$$\text{Work done} = Rx.$$

If then $P > R$, an amount of energy $(P - R)x$ remains to be accounted for. If on the other hand $P < R$, an amount of work $(R - P)x$ has been done at the expense of some source other than the effort P. Now we know from experience that, combined with the effects just stated, in the first case A will be moving faster at the end of the period considered than it was at the commencement, and in the second case slower at the end than at the commencement.

We are then inevitably led to the conclusion that—in the first case, the body has in some way stored up in

itself the amount of energy $(P-R)x$, this stored up energy showing its presence by the increased velocity; while in the second case the body is in some way capable of exerting energy to an amount $(R-P)x$, by having its velocity decreased. Taking the two cases together they lead to this one conclusion that—*a body in motion possesses, by virtue of its motion, a store of energy which can be added to or be drawn on.*

Kinetic Energy.—The amount of energy thus stored up in a body in motion is called the Kinetic Energy of the body.

We can now see what form the principle of work takes when applied to the cases just considered. For—

In the first case the body had, at the commencement, a certain amount of kinetic energy, this amount we call its Initial K. E. (K. E. being a common abbreviation for kinetic energy). But during the motion an amount $(P-R)x$ is added to the store of energy, therefore

$$\text{Final K. E.} = \text{Initial K. E.} + (P-R)x.,$$

or

$$Px = Rx + \text{Final K. E.} - \text{Initial K. E.},$$

and it can be easily seen that the second case gives exactly the same equation. Putting this equation in words, it is

Energy exerted = work done + change of K. E. of moving body,

which is the new form of the principle of work.

It must be noticed that *change* here means Final – Initial, which may be either positive or negative. We have given the form above since it is the one usually given, but the student will find it advisable, certainly on a first study, and probably even in all cases, to use the more extended form—

Energy exerted = work done + Final K. E. – Initial K. E.

subdividing again, if necessary, work done into useful and waste, *i.e.* if friction be taken into account.

We have seen now that a body in motion has K. E., but before we can make use of this we require to know how much energy a given body moving at a given velocity possesses, in virtue of this velocity.

The question here put can only be answered by experiment; or, at least, by using those laws of motion which represent the results of innumerable experiments. One case and one only we can settle at once, viz. when the body is still: it is then, we know, incapable of doing any work by altering its motion, so it has no available store of energy, or its K. E. is zero.

To determine the relation between velocity and K. E., we will take a simple familiar case, viz. a body moving under the action of gravity only.

Let the body fall from a state of rest at a height h ft. Then when it reaches the ground it will have attained a velocity v f.s., which is, we know, given by

$$v^2 = 2gh \text{ where } g = 32.2.$$

[The value of g varies at different points on the earth, g f.s. being the velocity of a freely falling body at the end of one second from rest, and since this velocity varies the number g also varies. Its variation is not, however, of practical importance.]

Apply now the principle of work.

Then we have

$$\begin{aligned} \text{Effort} &= \text{pull of gravity on the body,} \\ &= \text{weight of body} = \text{W lbs. say,} \\ \text{Resistance} &= 0, \\ \therefore \text{W}h &= 0 + \text{Final K. E.} - \text{Initial K. E.} \end{aligned}$$

Also

$$h = \frac{v^2}{2g}, \text{ and Initial K. E. is zero,}$$

$$\therefore \text{Final K. E.} = \frac{\text{W}v^2}{2g} \text{ ft.-lbs.}$$

The K. E., then, of a body which has fallen till its velocity is v f.s., is $\text{W}v^2/2g$ ft.-lbs.

But now it cannot matter how the body has arrived at its velocity v. For, taking two bodies, one of which has fallen from rest as above; while the other has been through any number of different changes of velocity, finally arriving at the downward velocity v; it is plainly impossible to detect any difference in the two bodies, or to conceive that one could give out more energy than the other. Hence we can leave out the falling from rest, and state simply that a body falling with velocity v f.s. has $Wv^2/2g$ ft.-lbs. of K. E.

We will now go further still. For the equation $h = v^2/2g$ does not require that the force W lbs. act through a vertical distance of h ft; but let that force act on the body in ***any direction*** through a distance h ft., then a velocity $v = \sqrt{2gh}$ will be obtained.

It follows then from the above considerations that the K. E. of a body weighing W lbs., moving in any direction at v f.s., is $Wv^2/2g$ ft.-lbs.

Sliding Pairs.—We have now the means of examining in detail those processes which, so far, we have only been able to examine for certain periods, *e.g.* the steam engine for whole revolutions.

Take first the simple sliding pair with which we first commenced. Let W be weight of the slider, and the resistance be that due to friction, also let the effort be applied by a cord attached to a weight P.

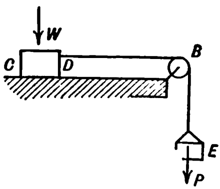


Fig. 143.

This is in a simple form the apparatus used by Morin for determining coefficients of friction (chap. ii.)

In describing the friction experiments, P was described as just sufficient to keep W moving; but this was not the case in all the experiments; and the motion we will examine is that which takes place when P is more than enough to keep W moving at a constant velocity.

Now we must first ask, to what system of pieces are we going to apply the principle of work? Is it to the single pair, W and the table; or to the two pairs—W and the table and the sliding pair P relative to the earth—together?

We will answer, to the two pairs together; or, in other words, we are going to consider the motion of the whole machine. We shall, however, neglect the weight of the cord and friction of the pulley.

Let us now consider a motion, starting from rest.

Resistance $=$ R, and is constant.

Let $x =$ distance moved by W, and therefore also by P. Then

$$\text{Energy exerted} = Px.$$
$$\text{Work done (all waste)} = Rx.$$
$$\therefore\ Px = Rx + \text{Final K. E.} - \text{Initial K. E.}$$

Now we have to consider—What do we mean by Final K. E.? Do we mean the K. E. of W, or the K. E. of P, or the sum of the two. It is in order to introduce this point that we have considered the motion of the two pieces P and W.

Taking the system as a whole, its K. E. is the sum of the separate K.E.'s of P and W; this we can easily see, for we could bring P to rest by applying to it some resistance, thus causing its K. E. to perform work against the resistance; but this would still leave W with the whole of its K. E., since it will go on irrespective of the stopping of P. The total K. E. of the system is then the sum of the K.E.'s of its parts, each reckoned separately, quite irrespective of what directions the parts may be moving in.

The foregoing statement applies to all systems of bodies, and we see then that when applying the principle of work to a machine, we must add together the K. E.'s of *all* the parts, and not only consider the K. E. of the body which supplies the resistance. At the end of

a given motion some parts may be moving faster than before, and they will have abstracted energy from that exerted by the effort; while others may be moving slower than at first, and those will have given up some of their store of K. E., thus aiding the effort. And the effect, we must thoroughly understand, is independent of direction, *i.e.* there is no such a thing as negative energy: taking a weight W, to give it a velocity of v f.s. to the right say, we must exert on it energy $Wv^2/2g$; also to give an equal weight W a velocity v to the left, we must exert again $Wv^2/2g$, so that, taking the two weights together, their total K. E. is $2Wv^2/2g$, in spite of the fact that they are moving in opposite directions, and thus we might say that, taken as a whole, the system has no velocity, since its centre of gravity does not move.

Returning now to our particular problem, let

v = final velocity of W, and also necessarily of P.

Then

$$\text{Final K. E.} = \frac{Wv^2}{2g} + \frac{Pv^2}{2g},$$

$$\therefore\ Px = Rx + (P+W)\frac{v^2}{2g},$$

$$\therefore\ \frac{v^2}{2g} = \frac{P-R}{P+W}\cdot x.$$

If then R be truly constant, this motion is what we may call a reduced copy of that of a freely falling body, viz. $v^2/2g = x$.

In Morin's experiments he found that the motion was such a reduced copy, and hence he drew the conclusion that R was constant during the motion, so that the ratio of R/W or f (page 51) was independent of the velocity, since the velocity varies during this motion while R does not.

Next consider the motion of a train. At starting Effort < Resistance, and the speed is accelerated till the

required speed of running, say v f.s., is reached. During this period let

P = pull of locomotive on the front carriage (supposed constant),
R = total resistance to carriages,
W = weight of train, *excluding engine.*

Then, considering the motion of the carriages,

$$Px = Rx + \frac{Wv^2}{2g} - 0$$

gives us the distance x that will be run, on a level, before speed is got up.

Still considering a level, the train will now run steadily at this speed; steam being partly shut off, since less effort is required.

Now to stop (neglecting for the present the action of brakes) the steam must be shut off entirely at a distance y from the stopping-place, given by

$$0 = R'y + 0 - \frac{W'v^2}{2g}, \quad R' = \text{total resistance to } \textit{whole} \text{ train.}$$

$$\therefore y = \frac{W'v^2}{2gR}, \quad W' = \text{weight of } \textit{whole} \text{ train.}$$

The student will very probably feel inclined to shorten the work by writing down at once

$$\frac{W'v^2}{2g} = R'y,$$

but this should be avoided as leading in many cases to mistakes, and to the confusion of Kinetic Energy with energy exerted.

In all cases we have one or more bodies exerting energy, a body or set of bodies which is acted on and which we may term the passive body, and one or more resisting bodies. Then in addition the passive body may have within itself resistances, which we may call internal.

Thus in the starting process we had: Energy-exerting

body—the engine; passive body—the carriages; resisting bodies—the rails and air.

Then there is an internal resistance at each axle; the effect of this is added to that of the rails and air, and the whole termed the Resistance.

But now in the stopping process we have plainly taken as our passive body the train, including the engine, or, as we call it, the *whole* train. Why have we made this difference in the two cases?

The reason is as follows: The real energy or effort-exerting body is the steam; but if in the first case we had considered this, we should have required to bring in the steam pressure, piston area, stroke, etc.; therefore for simplicity we took the pull on the draw bar as given, this pull being an effort exerted by the engine, its real source being of course the steam pressure. But in the stopping process, the simpler equation is got by taking the steam as the energy-exerting body; since steam being cut off, we obtain at once

$$\text{Energy exerted} = 0,$$

and hence our equation.

What now would have been the form of our equation had we kept to the same method for the stopping as for the starting? The engine is now an outside body, and having a store of K. E. it can exert energy on the train of carriages. Hence

$$\text{Energy exerted by engine} = \mathrm{R}y + 0 - \frac{\mathrm{W}v^2}{2g}.$$

When we come to ask how much energy the engine can exert, we must consider what its store, *i.e.* its K. E. is. The amount of energy possessed by the engine is $\mathrm{W}_1 v^2/2g$, W_1 being its weight. But it cannot exert all this on the carriages, because from this source must come the energy to overcome its own resistance, say R_1; this will absorb $\mathrm{R}_1 y$ of energy, leaving available

$$\frac{W_1 v^2}{2g} - R_1 y.$$

Hence this is the energy exerted by the engine on the carriages, and we have

$$\frac{W_1 v^2}{2g} - R_1 y = Ry + 0 - \frac{Wv^2}{2g},$$

$$\therefore (R + R_1) y = (W_1 + W)\frac{v^2}{2g},$$

but

$$R + R_1 \text{ is } R', \text{ and } W_1 + W \text{ is } W',$$

so we obtain as before

$$R'y = W'\frac{v^2}{2g}.$$

We see then that K. E. can appear as energy exerted, but it is energy exerted on *some other body*, *e.g.* the K. E. of the engine appears as energy exerted on the carriages. But in no case must we ever reckon the K. E. of a body as energy exerted on *itself*.

The method of stopping, which we have been considering, is, we know, not that actually employed. Actually the train keeps up full speed till at a much less distance than y from the station, and the stoppage is then effected by applying the brakes. These are blocks of wood, which are pressed against the rotating wheels, thus producing friction which increases the total resistance from R' to say R''. Then we have as before, if y'' be the new distance,

$$\frac{W'v^2}{2g} = R''y'',$$

and y'' will be much less than y was. The method of stopping here described is very wasteful of energy, since it causes the K. E. to be entirely dissipated as waste work. This is especially disadvantageous in cases where horses furnish the energy; the wear and tear of stopping and starting omnibuses and tramcars far exceeding that caused simply by keeping the speed up

when going. It has thus been an object to avoid, if possible, this loss; and one method which appears to have been fairly successful is to apply to the axle a strong spring, which is put in gear during stopping, and by its resistance to bending stops the car. It is then locked in place, and released when the car is to start; and it then, by unbending, restores to the car nearly the whole of the energy which it abstracted during stopping. If there were no losses, it would restore the whole of the original K. E., *i.e.* it would by itself get up the original velocity, but actually it does not quite do so, although it considerably relieves the horses.

The Bull Engine.—In many collieries the pumping is effected by an engine, the cylinder of which stands directly over the shaft, and the pump rods are attached directly to the piston.

Fig. 144 shows the arrangement diagrammatically, of course in actual cases the rods are much longer than here shown. Since there is no crank shaft, the stroke of the engine would appear to be limited only by the piston coming in contact with the cylinder ends. Actually, however, the stroke is governed by adjustment of the forces acting on the sliding piece, viz., piston and plunger rods, and we will now examine the question, using the graphic method.

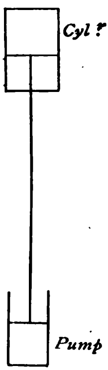


Fig. 144.

The pump is single acting, all the pumping being done on the down-stroke. During this stroke, or at least the greater part of it, the two ends of the cylinder are in free communication, so there is no effort due to steam pressure, the pressure simply balancing on the two sides of the piston. The effort then during the stroke is simply the weight of piston and rods, or as it is called the *pitwork*. The weight of the pitwork is rather more than neces-

sary to overcome the resistance of the water to the plunger, so that the whole descends at an increasing velocity. At a certain point of the stroke, however, the valve closes, shutting up the steam in the lower end of the cylinder, and then this steam acts as a buffer, gradually bringing the piston to rest, and preventing it from striking the cylinder end. We cannot, however, consider the actions during this down-stroke with any degree of accuracy without considering the acceleration of the water in the pipes, and the resistances

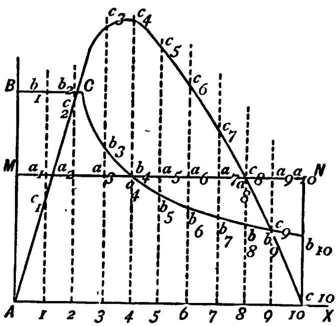


Fig. 145.

offered by these pipes, which questions require considerable knowledge of Hydraulics. We will then confine our attention to the up-stroke. Let AB represent the effective initial total pressure on the piston, this remains constant while the piston moves the distance BC; the steam is then cut off, and the pressure falls as the steam expands as shown by the curve Cb_{10}. We wish now to find how far the piston will move before its weight brings it to rest, or actually we require the converse result, viz., where should the cut-off take place so that the piston may not reach the top of the cylinder so as to strike against it, but may stop at a convenient

distance from it. The first is the simpler way of approaching the problem, and we can then see how to answer the second.

Take now AM to represent W, the weight of the pit-work, and draw MN parallel to AX. Then MN is the curve of resistance, and BCb_{10} the curve of effort.

Take now a point 1, on AX, representing the piston position when it has moved a distance A1 from rest. Draw $1a_1b_1$ vertical. Then, during the motion from A to 1 the energy exerted is represented by the area between the curve of effort and the ordinates at A and 1 (page 80),

$$\therefore \text{ Energy exerted} = \text{rectangle } BA1b_1.$$

Similarly,

$$\text{Work done} = \text{rectangle } MA1a_1.$$

But

$$\text{Energy exerted} = \text{Work done} + \text{Final K. E.} - \text{Initial K. E.},$$
$$\therefore \ BA1b_1 = MA1a_1 + \text{Final K. E.} - 0.$$

Therefore, if v_1 be the velocity of the piston,

$$\frac{Wv_1^2}{2g} = MBb_1a_1.$$

Let us now measure MBb_1a_1 in square inches, and set up $1c_1$ to represent it, so that if MBb_1a_1 is x sq. inches, $1c_1$ is x inches.

[The figure is for clearness drawn on a much larger scale.]

Then, on a scale to be hereafter determined, $1c_1$ represents the K. E. of the piston and rods. Continue this construction for points 2, 3, and 4; we have chosen 4 so as to be the intersection of MN and CD.

Thus we get the points c_1, c_2, c_3, and c_4, and we draw a curve through them. Then the ordinates of this curve represent at each point the K. E. of the piston, etc., when it reaches the point, so it is a curve of K. E. Its shape is easily seen to be a straight line up

to the point of cut-off; for up to that point the area of the rectangle MBb_1a_1 is directly proportional to Ma_1 or A1, so that $1c_1$ is proportional to A1. The K. E. then increases at a constant rate up to cut-off; after cut-off the excess of effort over resistance becomes less, so that the rate of increase of K. E. is not so great, and this is shown by the curve beginning to curve downwards, *e.g.* between 2 and 3 the gain of K. E. is $a_2b_2b_3a_3$, which is less than the gain between 1 and 2, viz., $a_1b_1b_2a_2$. It must be particularly noticed, however, that between A and 4, that is so long as Effort > Resistance, the K. E. continuously increases, and with it of course the velocity, since K. E. varies as v^2. A very common error is to imagine that because after C the effort falls off, the velocity must do likewise. The mistake lies in forgetting that the question whether the velocity will increase or not depends not on how much the effort exceeds the resistance, but on whether it exceed it at all. What is affected is, as we have seen above, the rate of increase.

At 4 the effort and resistance just balance, so that there is for the moment no increase of K. E., or the rate of increase has fallen to zero.

Next, we come to points past 4. And now the conditions have changed, for we have Effort < Resistance. Consider now the point 5. Then, for the whole motion from A,

$$\begin{aligned} \text{Energy exerted} &= ABCa_4b_55, \\ \text{Work done} &= Ma_55A, \\ \therefore\ \text{K. E. at } 5 &= ABCa_4b_55 - Ma_55A, \\ &= MBCa_4 - a_4a_5b_5. \end{aligned}$$

The ordinate $5.c_5$ will be less than $4.c_4$, which represents $MBCa_4$, by the area $a_4a_5b_5$, so that for points to the right of 4 we obtain the heights of the ordinates by subtracting from $4c_4$, the lengths representing $a_4a_5b_5$, $a_4a_6b_6$, etc. On the right of 4 then the curve begins to fall, and it will finally reach the axis AX at some point, which in our figure is marked 10, so c_{10} and 10 coincide.

We can easily see where 10 will be, for it is such that subtracting $a_4a_{10}b_{10}$ from $MBCa_4$ leaves nothing, so to find 10 we make

$$a_4a_{10}b_{10} = MBCa_4,$$

c_{10} being on the base line, shows that 10 is a point of zero K. E., so that 10 is the point at which the piston stops.

We now see what we have to do to find the cut-off when the stopping point is given. For given A10 we have, by trial, to find C, such that

$$MBCa_4 = a_4a_{10}b_{10}.$$

And then the piston will stop at the required point.

The curve Ac_1c_2 . . . 10 is a curve of K. E., but we have still to determine its scale.

Let now the scale of AB be 1 inch to m lbs., and that of A10, which represents the stroke, be 1 inch to n feet.

Then areas on the diagram represent energy on a scale 1 sq. inch to mn ft.-lbs. (page 70). The ordinate of the curve is, in linear inches, equal to the area on the diagram in sq. ins. (page 208), therefore the scale of the curve is

$$1 \text{ inch} = mn \text{ ft.-lbs.}$$

The curve is on this scale a curve of $Wv^2/2g$, but $W/2g$ is a constant, so that the curve will on a proper scale be also a curve of v^2. We can see what this scale is; for a line of 1 inch represents mn ft.-lbs., so that when the ordinate is 1 inch

$$\frac{Wv^2}{2g} = mn \text{ ft.-lbs.}$$

At that instant then $v^2 = 2gmn/W$, v being in f.s. If then we measure the ordinate on a scale of

$$1 \text{ inch} = \frac{2g}{W} mn \text{ feet},$$

the square root of this number of feet will give us the

value of v; we may say then the scale is 1 inch to $2gmn/W$ f.s. squared.

A further graphical construction can be used which will give a curve of velocity, but into this we have not space to inquire.

Turning Pairs—K. E. of Rotation.—When a body moves so that all the particles move in parallel straight lines, they all have necessarily the same velocity v. In this case a small particle of weight w has K. E. $wv^2/2g$, and the whole body a K. E. $Wv^2/2g$, W being the total weight. All the cases we have so far considered have been of this type.

But now, when a body is rotating, no two particles have the same motion, thus (Fig. 146) a particle at B has a velocity Ar_B, A being the angular velocity about O, the centre of rotation; and one at C has a velocity Ar_C. The velocities are respectively at right angles to OB and OC, so that even if $r_C = r_B$, the directions of motion are different, although the amounts of the velocities are then equal.

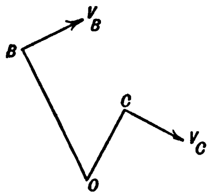


Fig. 146.

Each heavy particle of which the body is composed has then its own K. E., and since we have seen that K. E. is independent of direction, it follows that the total K. E. of a rotating body is the arithmetic sum of the K. E.'s of all the particles composing it, irrespective of the direction of their motions.

The calculation of this quantity, then, requires that we divide the body up into indefinitely small particles, and sum up the K. E.'s of all these particles. This process involves the use of the Integral Calculus, and hence the student must, until he has mastered the use of this, accept the results given as facts.

Thin Ring.—One case we can treat, viz., that of a thin ring or cylinder, rotating about its axis. Let r be

the mean radius. Then, if the thickness be very small compared to r, all the particles are practically at the same distance r from the centre. If then

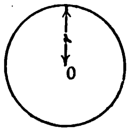


Fig. 147.

$$A = \text{angular velocity},$$
$$w = \text{weight of one particle},$$
$$v = \text{velocity of one particle},$$

we have

$$\text{K. E. of particle} = \frac{wv^2}{2g} = \frac{wA^2r^2}{2g},$$

and each particle will have the same K. E., so that

$$\text{Total K. E.} = W\frac{A^2r^2}{2g},$$

W being the weight of the ring.

This result can be used without serious error for most fly-wheels, W being the weight of rim, r its mean radius, and an allowance of two or three per cent added for the K. E. of boss and arms.

Radius of Gyration.—Consider now the case of an actual fly-wheel with a definite thickness of rim. Let R and R′ be the inner and outer radii. Take now a radius r intermediate between R and R′, and consider the particles which lie on a very thin ring of radius r. Then this ring has a K. E. due to the velocity Ar. All the particles outside this ring have velocities greater than Ar, and so their total K. E. will be greater than if they all lay on the ring; while, on the other hand, those which lie inside have velocities less than Ar, and their total K. E. is less than if they lay on the ring.

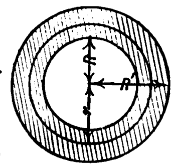


Fig. 148.

Comparing now the actual K. E. of the wheel, with what it would be if all the particles were concentrated on the ring of radius r, it appears there is an excess due to the particles outside the ring, and loss due to those

inside. It is not difficult then to see that by selecting r properly, we can make the gain and loss balance, and if this be done, then *the actual K. E. of the wheel is the same as if the whole mass were concentrated on a circle of radius r.*

The value of r which satisfies the above condition is called the **Radius of Gyration**, and when by use of the calculus the value of the radius of gyration is found, then calling it r, the K. E. of the rotating body is $WA^2r^2/2g$, where A is the angular velocity.

Although we cannot calculate the value of the radius of gyration without the calculus, yet we can show the method of proceeding, and this we will now do.

We will take the fly-wheel already considered. Divide it up into a very large number of thin rings.

Let their radii be in order

$$r_1, r_2, r_3, \ldots, r_n,$$

r_1 being very nearly R, and r_n nearly R′.

Let the weights of the rings be

$$w_1, w_2, w_3, \ldots, w_n.$$

Then

$$\text{K. E. of 1st ring} = w_1\frac{A^2r_1^2}{2g},$$

$$\text{K. E. of 2d ring} = w_2\frac{A^2r_2^2}{2g},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\text{K. E. of nth ring} = w_n\frac{A^2r_n^2}{2g},$$

$$\therefore \text{Total K. E.} = \frac{A^2}{2g}\{w_1r_1^2 + w_2r_2^2 + w_3r_3^2 + \ldots + w_nr_n^2\}.$$

It is for the summation of the quantity in brackets, when n is indefinitely large, that we require to use the calculus. We shall meet a similar expression further on in the book, and it is known as the **Moment of Inertia.**

But if r be the radius of gyration,

$$\text{Total K. E.} = \frac{WA^2r^2}{2g}, \quad \text{or} \quad \frac{(w_1+w_2+\ .\ .\ .\ +w_n)A^2r^2}{2g}$$

Hence equating the two values of the K. E.,

$$r^2 = \frac{w_1r_1^2+w_2r_2^2+\ .\ .\ .\ +w_nr_n^2}{w_1+w_2+w_3+\ .\ .\ .\ +w_n}.$$

The results of this calculation for some particular cases are as follows :—

Solid cylinder of radius a rotating round its axis	$r^2=a^2/2$
A rod, length l, rotating round an axis through its end, perpendicular to its length . .	$r^2=l^2/3$
Parallelogram of height h rotating round its base	$r^2=h^2/3$
Do. about a centre line parallel to its base .	$r^2=h^2/12$
A circular plate, radius a, diameter h, rotating about a diameter	$r^2=a^2/4=h^2/16$
A triangle, height h, about its base . . .	$r^2=h^2/6$
Do., about an axis through its CG, parallel to its base	$r^2=h^2/18$
Do., about an axis through its vertex, parallel to its base	$r^2=h^2/2$

Most of these results are not required for the present, but they are collected for future reference.

We are now in a position to calculate accurately the K. E. of the rim of a fly-wheel, and so see how near our approximate result would be.

We proceed thus—

Let R_1 and R_2 be the inner and outer radii, then

$$\text{K. E. of wheel} = \text{K. E. of cylinder radius } R_1 - \text{K. E. of cylinder radius } R_2.$$

Let W_1 and W_2 be the weights of these cylinders, then

$$\text{Weight of wheel } W = W_1 - W_2,$$

$$= W_1 - W_1\frac{R_2^2}{R_1^2} = \frac{R_1^2 - R_2^2}{R_1^2}W_1 \quad (1),$$

$$\begin{aligned}\therefore \text{K. E.} &= \left(\frac{W_1R_1^2}{2} - \frac{W_2R_2^2}{2}\right)\frac{A^2}{2g},\\ &= \left(\frac{W_1R_1^2}{2} - W_1\frac{R_2^2}{R_1^2} \cdot \frac{R_2^2}{2}\right)\frac{A^2}{2g},\\ &= \frac{W_1}{2} \cdot \frac{R_1^4 - R_2^4}{R_1^2} \cdot \frac{A^2}{2g},\\ &= \frac{W_1}{2} \cdot \frac{R_1^2 - R_2^2}{R_1^2} \cdot \left(R_1^2 + R_2^2\right)\frac{A^2}{2g},\\ &= \frac{WA^2}{2g} \cdot \frac{R_1^2 + R_2^2}{2} \quad \text{[from (1)].}\end{aligned}$$

Our approximate result was

$$\text{K. E.} = \frac{WA^2}{2g}\left(\frac{R_1 + R_2}{2}\right)^2,$$

so the error is

$$\frac{WA^2}{2g}\left\{\frac{R_1^2 + R_2^2}{2} - \frac{R_1^2 + 2R_1R_2 + R_2^2}{4}\right\} = \frac{WA^2}{2g} \cdot \frac{(R_1 - R_2)^2}{4},$$

which decreases as R_2 approaches R_1.

This method of finding the K. E. of a body, by considering it as the difference of two other bodies, or in other cases as their sum, can be often used, but we must be careful to take the K. E.'s all about the one given axis.

Motion of an Unconstrained Body.—The principle of work enables us to solve questions relating to the motion of an unconstrained body, *i.e.* a body whose motion is not defined by its connection to other bodies, *e.g.* a shot after leaving the gun.

This case is of interest, and we will briefly examine it.

The shot lies in the gun, and the powder burning behind it causes a gaseous pressure which forces it along the bore; during this period the motion of the shot is defined by the nature of the pairing between it and the bore of the gun. It now issues from the mouth of the gun with a certain velocity and K. E., and, during the remainder of its motion before reaching the earth or target, its motion is free from any constraint other

than that exerted by gravity. We have then the reverse of our usual problem, for generally we know the path of the body and require to determine the forces acting, while here we know the force and not the path. Let

$$W = \text{weight of shot,}$$
$$v_1 = \text{velocity on leaving the muzzle,}$$

and consider the motion between the moment of leaving the muzzle and that when the shot has attained a height h. Then

$$0 = Wh + \frac{Wv^2}{2g} - \frac{Wv_1^2}{2g},$$

v being the velocity the shot has at the moment the height h is reached.

This example shows well the importance of clear definition of the period of time chosen (page 137). For suppose we take the period to be from the moment of igniting the powder. Then we have

$$\text{Energy exerted} = \text{whole energy of powder.}$$

Then, assuming for simplicity that the bore of the gun offers no resistance to the shot, we have

$$\text{Energy of powder} = Wh + \frac{Wv^2}{2g} - 0.$$

Comparing this with the preceding we see that we must have

$$\text{Energy of powder} = \frac{Wv_1^2}{2g},$$

and this is correct, for this equation is the form taken by the principle of work for the period during which the shot is being pushed along the bore, since we then have

$$\text{Energy exerted} = \text{energy of powder,}$$
$$\text{Work done} = 0,$$
$$\text{Initial K. E.} = 0, \quad \text{and Final K. E.} = \frac{Wv_1^2}{2g}.$$

If the bore offer a resistance R and its length be l,

then the work done will be increased in the two preceding equations by the term Rl, and the K. E. $Wv_1^2/2g$ will be accordingly diminished.

Potential Energy.—Returning to the consideration of the motion when unconstrained, the principle of work is sometimes stated in a new form.

The only resistance is gravity (neglecting the friction of the air), and this is a reversible resistance (page 41). The work done against gravity during the rise h, viz. Wh, will be restored again when the shot falls. This fact is sometimes expressed by saying that the shot has Potential Energy Wh, due to being at a height h. The equation

$$0 = Wh + \frac{Wv^2}{2g} - \frac{Wv_1^2}{2g}$$

then becomes

$$0 = \text{Final Pot. E} + \text{Final K. E.} - \text{Initial K. E.},$$

or, remembering that the initial potential energy is zero, and that the equation refers to any point during the flight, since h may have any value, we have at any point of the flight

$$\text{Pot. E.} + \text{K. E.} = \text{Initial Pot. E.} + \text{Initial K. E.},$$

so that the sum of the potential and kinetic energies is constant for all points of the flight.

The principle of work may then be written for this case as

$$\text{Potential Energy} + \text{Kinetic Energy} = \text{constant}.$$

EXAMPLES.

1. A slider weighing 100 lbs. rests on a table, it is moved as in Fig. 143 by a weight of 20 lbs., and when it has moved 2 ft. its velocity is observed to be 2 f.s. Find the coefficient of friction.

Ans. .163.

2. In question 1, page 90, the rider goes 60 yds. from rest before getting the speed up. Find the mean moment he exerts.

Ans. 34.3 lbs.-ft.

3. A train weighs 60 tons and the engine 25 tons. It is ascending an incline of 1 in 100 at 30 miles per hour when the

draw bar breaks. Taking the resistance at 16 lbs. per ton, find how far the carriages will run before stopping; also what speed would the engine finally attain if it continued to exert the same H. P. as before the breakage, and its resistance were unaltered? *Ans.* $\frac{1}{3}$ mile; 102 miles per hour.

4. If in the preceding the brakes did not act, what would be the speed of the train when running back and passing the point at which the breakage occurred?

Ans. $12\frac{1}{2}$ miles per hour.

5. In question 3, the guard's van weighs 18 tons, and he applies his brake, skidding the wheels, directly the breakage occurs. Find in what distance the train is brought up. Coefficient of friction between wheel and rail .18. *Ans.* $\frac{1}{17}$ mile.

6. The piston and pump rods of a "Bull" engine weigh 18 lbs. per sq. in. of piston. The initial steam pressure is 50 lbs. absolute, and the cut-off takes place when the piston has travelled one foot of the up-stroke. Assuming hyperbolic expansion, find the least length of cylinder, that the piston may not strike the cover. Back pressure 2 lbs. *Ans.* 7 ft. 7 ins.

7. In the preceding find the position of maximum piston velocity, and the corresponding K. E. of the moving parts per sq. in. of piston.

Ans. 2 ft. 6 ins. from commencement; $45\frac{3}{4}$ ft.-lbs.

8. A body weighing 112 lbs. is fastened to a rope passing over an axle 2 ins. diameter, on which is a fly-wheel 2 ft. diameter. Find the weight of the fly-wheel rim, so that the body after falling 40 ft. may have a velocity of only 4 f.s.

Ans. 124 lbs.

9. Find the K. E. of a disc running at a speed V f.s. along a plane. Radius r ft.

Ans. The disc is moving as a whole at velocity V f.s., and also rotating with an angular velocity V/r. The K. E. is the sum of that due to each motion separately, which can be proved analytically, or may be seen as follows: Suppose the disc to have a loose axle through its centre, then by stopping this axle we can take out the K. E. $Wv^2/2g$; but the disc will then still rotate about the axle with its original angular velocity; hence the result.

$$\therefore \text{Total K. E.} = \frac{Wv^2}{2g} + \frac{WA^2a^2}{2g},$$
$$= \frac{3}{2}\,\frac{Wv^2}{2g}.$$

The separate K. E.'s are said to be Translation K. E. and Rotation K. E. respectively.

10. A disc and hoop are running at the same speed on the level, and commence to ascend an incline. Which will ascend higher, and by how much?

Ans. The hoop $\frac{4}{3}$ as far as the disc.

11. Determine the weight of fly-wheel rim per horse power, which, when running at 70 feet per second, will have stored in it 10 per cent of the energy exerted per minute.

Ans. 43 lbs.

12. The four wheels of a truck consist of solid discs of cast-iron 2 ft. diameter and $1\frac{1}{2}$ in. thick. The truck, when moving with a velocity of 20 miles an hour, commences to ascend an incline of 1 in 100. The weight of the truck, with wheels, is half a ton, and the resistance 10 lbs. per ton. Find how far the truck will run up the incline. *Ans.* 1220 ft.

13. In the motion of a projectile the horizontal component of the velocity remains constant. Hence deduce by means of the principle of work the greatest height to which a shot, projected with initial velocity V at an elevation α° will rise.

Ans.

$$\text{Minimum K. E.} = \frac{W(v\cos\alpha)^2}{2g},$$

$$\therefore \text{Maximum Potential Energy} = \frac{Wv^2\sin^2\alpha}{2g},$$

$$\therefore \text{Greatest height} = \frac{V^2\sin^2\alpha}{2g}.$$

14. A ship of 2500 tons displacement is propelled at 20 knots by engines of 8000 H. P.; estimate the distance which will be traversed by the ship whilst an amount of energy is developed by the engines equal to the K. E. stored in the ship.

Ans. 760 feet.

CHAPTER XI

THE DIRECT ACTOR—FLUCTUATION OF ENERGY AND SPEED

WE are now in a position to examine into that irregularity of motion which we saw (page 188) must occur owing to the irregularity of the driving effort, but into the magnitude of which we were not then able to inquire.

We shall suppose the piston pressure to be uniform, and the curve of crank effort will accordingly be that obtained by the method of chap. ix. Furthermore we shall examine only the case of an engine running at *constant speed.*

Before commencing our work we must devote some little space to the consideration of the meaning of this term "constant speed" as we have just used it. Plainly it cannot mean that all parts of the engine run at a constant velocity; because that, as we have seen, is an impossibility; even if the crank velocity were constant, the piston velocity could not be so. But it is not even necessary for "constant speed" that the crank velocity be constant. The practical meaning of the term is that each revolution is performed in the same time; but this does not prevent the velocity of the crank being different at different parts of the revolution. When an engine then is running at constant speed, the crank commences a revolution with a certain velocity; this velocity goes through a certain series of changes during the revolution, but ends up with the same value as it commenced with;

then the next revolution starts with the same velocity as the first, the same series of changes is gone through, and so on over and over again. Evidently what applies to one piece applies to all; so that each piece is going continuously through a certain series of changes of velocity. The time of revolution then will be constant, while the actual velocity need not be the same at any two instants of a revolution.

Periodic Motion.—The motion here described is one case of what is known as periodic motion. Periodic motion meaning a motion in which a certain set of changes of velocity is gone through over and over again. The time of completing a set is called the Period. If successive periods are all equal, we may call the motion Uniform Periodic, but usually the term periodic is taken to include equality of periods, and so implies uniform periodic.

The period for a steam engine is one revolution, that giving a complete set of changes. But it is not always one revolution that makes a period. Take, for example, a gas engine in which an explosion of gas takes place only in every alternate revolution. Then at the commencement of the explosion stroke the parts have certain velocities; the gas now explodes, driving the piston before it, and accelerating the velocity of the shaft; now the piston returns, driving out the products of the explosion, and a second revolution commences; but during this revolution no effort is applied; and so the velocity gradually decreases through the return half of the first and the whole of the second revolution, until it reaches, at the commencement of the third, exactly the same velocity as it had at the commencement of the first, and now the series commences over again. In this case then we have periodic motion, but the period is not one, but two revolutions. Evidently to any periodic motion, taking a whole period, the balanced forces principle of work applies (page 61).

So then for constant speed of an engine the only condition necessary is that during a whole revolution

$$\text{Mean resisting movement} = \text{mean driving movement,}$$
$$= \frac{2}{\pi}\mathrm{P}a \quad \text{(for a single cylinder).}$$

Generally, however, as we stated on page 188, the resisting moment is fairly uniform, and hence we will examine the case in which it actually is so. In that case its mean and actual values are the same, viz., $2\mathrm{P}a/\pi$, or we may consider a resistance $2\mathrm{P}/\pi$ applied to the crank pin.

Fluctuation of Energy and Speed.—We will now summarise the conditions of the engine we are

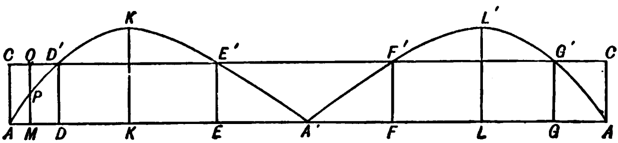


Fig. 149.

going to consider, viz. Constant Piston Pressure, Weightless Reciprocating Parts (piston, etc., see page 182), No Friction, Constant Resistance.

In Fig. 149 we have drawn a linear curve of crank effort for a single cylinder, and marked on it the line CC of mean crank effort. This line CC will now be the curve of resistance, as we have just shown.

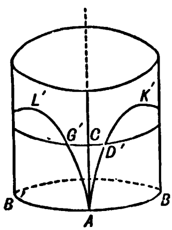


Fig. 150.

[It is important that the student should clearly comprehend that these curves of effort and resistance are drawn according to the conditions of chap. iv. page 79, so that their areas give energy exerted on the crank shaft, or work done against the resistance to the shaft. Some assistance to such clear comprehension may be obtained by drawing the curve on paper, and then cutting along the base AA, up the sides AC, and continuing along

the dotted lines in Fig. 149. Now roll up the strip so obtained by bringing A to A and C to C, so obtaining Fig. 150. It will then be seen that ABA′B′A is the actual crank circle, and that the efforts and resistance are set off at right angles to the path of the moving crank pin. We also see that the two ends AA, and also CC, really represent only one point A and one C respectively.]

We can now proceed by the same method as used in the consideration of the "Bull" engine in the last chapter.

Commencing at A.

The crank shaft and fly-wheel (if any) are rotating with an angular velocity, which we will denote by A_A.

At any point, as M, between A and D, the effort MP is less than the resistance MQ. The velocity will then continuously decrease until the crank pin reaches D. Calling the angular velocity at D, A_D; and taking

$$\text{Weight of shaft and fly-wheel} = W,$$
$$\text{Radius of gyration} = r,$$

we have, applying the principle of work to the movement from A to D,

$$\text{Energy exerted} = \text{area } AD'D,$$
$$\text{Work done} = \text{area } ACD'D,$$
$$\text{Final K. E.} = \frac{WA_D{}^2r^2}{2g}, \quad \text{Initial K. E.} = \frac{WA_A{}^2r^2}{2g},$$
$$\therefore\ ADD' = ACDD' + \frac{WA_D{}^2r^2}{2g} - \frac{WA_A{}^2r^2}{2g},$$
$$\therefore\ \frac{WA_D{}^2r^2}{2g} = \frac{WA_A{}^2r^2}{2g} - ACD'.$$

The change of K. E. then is $-ACD'$.

Now directly the pin passes D the effort is greater than the resistance, so the speed increases, and this holds till E is reached, where equality of effort and resistance is again reached.

Between D and E we have Effort > Resistance, and change of K. E.,

$$\frac{WA_E{}^2r^2}{2g} - \frac{WA_D{}^2r^2}{2g} = +D'K'E'.$$

Notice here that K, the point of *maximum* effort, is not one of maximum speed or K. E., this is a common mistake (compare page 209); the rate of increase increases up to K and is a maximum at K, so K is a point of maximum acceleration. Similar remarks apply to A′ etc. We need not go fully into the work, as we have already done so in the case of the Bull engine, and we will therefore put it as briefly as possible.

Between E and F. Effort < Resistance. Change of K. E.,

$$\frac{WA_F{}^2r^2}{2g} - \frac{WA_E{}^2r^2}{2g} = -E'A'F'.$$

Between F and G. Effort > Resistance,

$$\frac{WA_G{}^2r^2}{2g} - \frac{WA_F{}^2r^2}{2g} = +F'L'G'.$$

Finally G to A. Effort < Resistance,

$$\frac{WA_A{}^2r^2}{2g} - \frac{WA_G{}^2r^2}{2g} = -G'AC.$$

By beginning at the commencement of the stroke we have split one period of decreasing velocity—viz. from G through A to D—into two parts. Looking at the folded curve in Fig. 150, we see that the parts G′AC, D′AC really form one triangular piece G′AD′. Thus for the motion from G to D, change of K. E.,

$$\frac{WA_D{}^2r^2}{2g} - \frac{WA_G{}^2r^2}{2g} = -G'AD',$$

and we go on, for the next revolution, over again.

We have now found then, that at D the K. E. and velocity had been decreasing and commence to increase.

But these last words are the definition of a *minimum* value. D then is a point of minimum K. E. and velocity.

[Since K. E. varies as v^2 it is of course evident that K. E. and velocity increase and decrease together.]

At E both the K. E. and the velocity, which hitherto have been increasing, commence to diminish.

This is the definition of a *maximum* value. Hence E is a point of maximum K. E. and velocity.

Similarly at F we have a minimum, and at G a maximum.

There are then two points of maximum velocity and two of minimum.

We must not now confuse *maximum* and *minimum* with *greatest* and *least*. There can be only one greatest and one least; and it may even be that in some cases a minimum value is greater than a maximum one.

Take, for example, Fig. 151. Then AA, BB, CC are maximum ordinates of the curve; while DD, EE are minimum ordinates; yet the minimum DD is greater than the maximum CC. The greatest and least are AA and EE, *i.e.* the greatest maximum and the least minimum.

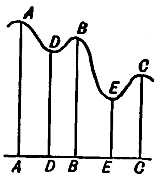


Fig. 151.

We can in our present case determine the positions of greatest and least velocity. For—

First we notice, that from D round to D again there are two gains of K. E., viz. D′K′E′ and F′L′G′; and two losses, E′A′F′ and G′AD′. And since we end up at D with the same velocity—*i.e.* same K. E.—as we commenced with, the gains and losses must balance,

$$\therefore \text{D}'\text{K}'\text{E}' + \text{F}'\text{L}'\text{G}' = \text{E}'\text{A}'\text{F}' + \text{G}'\text{AD}'.$$

Looking at Fig. 149 we see plainly that E′A′F′ is greater than G′AD′,—*i.e.* G′AC + ACD′,—while D′K′E′ and F′L′G′ are equal. Hence

$$\text{E}'\text{A}'\text{F}' > \text{D}'\text{K}'\text{E}' \text{ or } \text{F}'\text{L}'\text{G}',$$

and

$$\text{D}'\text{K}'\text{E}' \text{ or } \text{F}'\text{L}'\text{G}' > \text{G}'\text{AD}'.$$

Now there is a maximum at E and also at G.

Between E and G we lose (in K. E.) E′A′F′ but gain F′L′G′.

This gives on the whole a loss, so that the K. E. at E is greater than at G.

Therefore E is the position of greatest K. E. and velocity.

For the least we have minima at D and F. Between D and F the gain of K. E. is D′K′E′, and the loss E′A′F′.

There is then a loss in going from D to F; and therefore F is the position of least K. E. and velocity.

Thus we see that the amount of K. E. stored in the revolving parts of the engine is constantly fluctuating. Commencing at D, they gain D′K′E′, then lose E′A′F′, gain F′L′G′, and lose G′AD′. Each of these areas represents *a* **Fluctuation of Energy**, while the greatest of them —*i.e.* E′A′F′—is called **The Fluctuation of Energy.**

The amount of these fluctuations can be expressed either in foot-lbs. or by the ratio which they bear to the whole energy exerted by the engine in a revolution.

In the first case we calculate their areas by the methods of mensuration, and then deduce from the scale of the figure the amounts of energy which these areas represent (see page 71).

In the second method, since the whole energy exerted in a revolution is equal to the area of the effort curve AD′K′EA′F′L′G′A, or equally of the work curve ACCA; we find the ratio which these fluctuation areas bear to the area of the rectangle ACCA. We thus obtain four fractions, each of which is a **Coefficient of Fluctuation**, the greatest one is **The Coefficient of Fluctuation of Energy.**

Necessary Weight of Revolving Parts.—Given a constant piston pressure, and the ratio of connecting rod to crank, the shape of the crank effort curve is determined, and also the amount of fluctuation of energy, quite irrespective of the value of W. But the foregoing

does not hold with regard to the fluctuation of speed produced, as we shall now see.

Generally we require that the fluctuation of speed be kept within certain limits, depending on the purpose for which the engine is used; some purposes—*e.g.* electric lighting—requiring great uniformity, while for some kinds of rough work a variation of 50 per cent is not of much importance. The amount of fluctuation which can be allowed is given generally as a fraction of the mean speed, this fraction being called **The Coefficient of Fluctuation of Speed.**

Let now the coefficient of fluctuation of energy as determined from the diagram be k, and the coefficient of fluctuation of speed be c, c being given as one of the data, its determination for any case depending on considerations outside the scope of the present work.

Then if A_0 = mean angular velocity at which the engine is to run, the fluctuation of speed $= cA_0$.

But the fluctuation of speed is from A_E to A_F, these being the greatest and least respectively,

$$\therefore \ cA_0 = A_E - A_F.$$

Next, the mean speed is A_0. Now we cannot say that $(A_E + A_F)/2$ is the mean speed accurately; because to find the real mean speed, we ought to construct a curve of speed and then find its mean height (compare page 22). This mean height will not, in all probability, be equal to $(A_E + A_F)/2$, the mean of the greatest and least; but it will, in nearly all cases, be very nearly $(A_E + A_F)/2$, and it will be sufficiently accurate for our purpose if we take it to be so. This gives then

$$A_0 = \frac{A_E + A_F}{2}.$$

Let us represent the energy exerted by the steam in one revolution by E_0. Then between E and F we have

$$\frac{W A_E^2 r^2}{2g} - \frac{W A_F^2 r^2}{2g} = k E_o,$$
$$\therefore \frac{W r^2}{2g}(A_E^2 - A_F^2) = k E_o,$$
$$\therefore \frac{W r^2}{2g}(A_E - A_F)(A_E + A_F) = k E_o,$$
$$\therefore \frac{W r^2}{2g} c A_0 . 2 A_0 = k E_o,$$
$$\therefore \frac{W A_0^2 r^2}{2g} = \frac{k}{2c} E_o.$$

Putting the result here obtained in words, it is—*In order that the fluctuation of speed may not exceed c times the mean speed, it is necessary that the K. E. of the revolving parts, when running at the mean speed, should be k/2c times the energy exerted by the steam in one revolution.*

Fly-Wheels.—In these equations W is the weight of all the revolving parts, and r their radius of gyration. But in most cases the effect of the fly-wheel is so great compared with that of the remaining parts, that we treat the revolving parts as if they consisted of the fly-wheel only. In this case W will be the weight of fly-wheel required, and r its radius of gyration, which, as we have seen in the last chapter is, nearly enough, the mean radius of the rim.

Our equation above gives us only one relation between W and r, while to determine them definitely we require two. We can obtain a second relation, or at least an equation for r, as follows. It is of course advantageous to have W as small as we can, *i.e.* r as large as possible. But if V_0 be the mean speed of the rim

$$V_0 = A_0 r,$$

so that making r large makes V_0 large also. Now when a body revolves there is a tendency, due to centrifugal force, to burst (see chap. xii), this tendency increasing as the speed increases. This effect puts a

limit to the value of V_0, which is frequently taken as 80 ft. per sec. Thus A_0 being given, the value of r must be less than $80/A_0$, and we can then determine r and then W.

The Two-Crank Engine.—In a single engine a fly-wheel, or something equivalent, is an absolute necessity; we will now see how far this is the case in a two-crank engine. A brief statement will be sufficient, as full explanations have been already given.

Fig. 152 shows the linear curve of effort, lettered as usual.

We will commence at D, then, in order, the K. E.'s are—at E, a maximum; F, minimum; G, maximum; H, minimum; K, maximum; and D, a minimum.

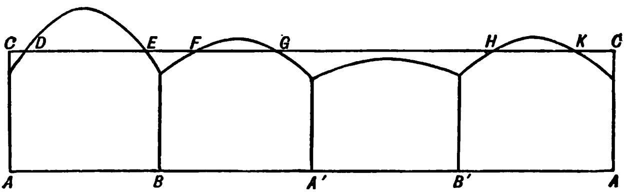


Fig. 152

In the actual figure drawn the fluctuations of energy are in order of size—1st, G to H; 2d, D to E; 3d and 4th equal, F to G and H to K; 5th and 6th equal, E to F and K to D.

Whence we deduce that the greatest maximum velocity is at G, and the least maximum at K; while the least minimum velocity is at H, and the greatest minimum at F.

The calculation of the numerical values we will leave, as an example, to the student.

We see at once from the figure that the value of k is much smaller than in the single engine, and hence a much less weight of fly-wheel is necessary to obtain a given regularity of speed. But besides this there is

another reason which tends to render the fitting of a fly-wheel to this type of engine unnecessary, and this we will now consider.

Effect of Reciprocating Parts.—We have so far considered these parts as weightless in the present chapter. But they are actually heavy, and will accordingly affect the motion, and we will now examine their effect.

In the case of the Bull engine (page 206) the reciprocating parts were very heavy, and their motion was determined simply by the relation between their weight, the varying effort, and the resistance. In our present case they are not nearly so heavy comparatively, and, moreover, their motion is determined to a very great extent simply by the connection between them and the rotating parts.

Let us, for simplicity, neglect the effect of obliquity. Then the piston, rod, and connecting rod of the leading crank, OP in the figure, all move forward at the instant with velocity

$$V_P \sin \theta.$$

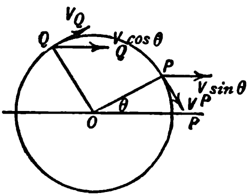


Fig. 153.

The following crank will be at OQ, and the velocity of its reciprocating parts will be

$$V_Q \cos \theta, \quad \text{or } V_P \cos \theta,$$

since necessarily $V_Q = V_P$.

The total K. E. then of the two sets of reciprocating pieces is

$$\frac{W'V_P{}^2 \sin^2 \theta}{2g} + \frac{W'V_P{}^2 \cos^2 \theta}{2g},$$

or

$$\frac{W'V_P{}^2}{2g} \quad \text{(W' being the weight of } \textit{one} \text{ set).}$$

The two sets together then have always the same K. E. as a fly-wheel of weight W′, mean radius of rim *a*,

would have, if such a wheel were fixed to the shaft. We say *always*, because OP is any crank position whatever.

Thus then, the motion of the shaft will be identical with what it would be if the reciprocating parts were weightless, and such a heavy fly-wheel as we have described were fixed to the shaft.

Hence the double crank engine is preferable to the single, not only by reason of the greater regularity of effort, but also by containing in itself an equivalent to a fly-wheel.

It does not follow, however, from what we have just said that it is necessarily advantageous to have heavy reciprocating parts. We have shown that such will be advantageous as regards regularity of motion of the engine as a whole; but it may very likely be that such regularity is obtained at the expense of great irregularity of force in the cylinders separately. We must therefore examine the effect on one cylinder separately.

Inertia of Reciprocating Parts.—To effect this we shall have, for simplicity, to make the assumption that the crank revolves uniformly.

[This assumption would of course be unwarranted if we were attacking the question for the first time, since the motion of the crank would be one of the results to be obtained. But we know already that the crank does rotate very nearly uniformly, and hence we make the assumption, knowing the error caused will be only slight.]

We will also neglect obliquity.

Suppose now that the crank is rotating uniformly; then before the steam pressure on the piston can produce any effort on the crank, it must first keep the reciprocating parts up to the crank pin. For example, let us start the stroke with the crank pin accurately centred in its bearing (Fig. 154), and therefore, since there is usually some clearance, not touching it anywhere.

[We must of course suppose the mere dead weight of the rod kept off the bearings in some way.]

Now as the crank revolves uniformly, a certain steam

pressure is required to keep the rod up to and centred on the pin, quite irrespective of any effort being exerted

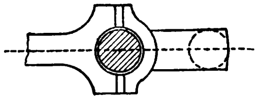


Fig. 154.

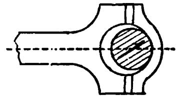


Fig. 155.

on the pin; and if this pressure be not supplied, the pin will drag the rods with it, the pin bearing as in Fig. 155. If just the necessary pressure be applied the pin keeps centred, as in Fig. 156, while if more than this be applied, the rod drives the pin (Fig. 157), but the effort

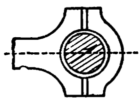


Fig. 156.

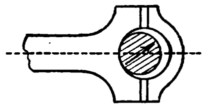


Fig. 157.

it exerts is only due to the excess of the actual pressure over that required to keep the bearing centred relative to the pin.

[In the figures the clearance is much exaggerated for clearness. On the scale of the figures it would not actually be visible, but can seldom be entirely absent.

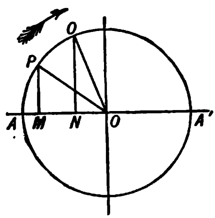


Fig. 158.

We propose now to find what pressure is necessary to keep the rods up to the crank.

Consider the motion OP to OQ. Let

$$<AOP=\theta_1, \quad AOQ=\theta_2,$$
$$OM=x_1, \quad ON=x_2,$$
$$W'=\text{weight of reciprocating parts.}$$

Then their velocity is at P

$$V_o \sin\theta_1 = V_o \cdot \frac{PM}{a},$$

and at Q

$$V_o \sin\theta_2 = V_o \cdot \frac{QN}{a}.$$

There is no work done because the pin is not touched by the rod, or because that is one of our conditions,

$\therefore$ Energy exerted by steam = Final K. E. − Initial K. E.,

$$=\frac{W'.V_o^2}{2g}\cdot\frac{QN^2}{a^2}-\frac{W'V_o^2}{2g}\cdot\frac{PM^2}{a^2}.$$

Let now P = mean pressure during the motion. Then

$$\text{Energy exerted} = P(x_1 - x_2),$$

$$\therefore\ P(x_1-x_2)=\frac{W'V_o^2}{2ga^2}[a^2-x_2^2-(a^2-x_1^2)],$$

$$=\frac{W'V_o^2}{2ga^2}(x_1^2-x_2^2),$$

$$\therefore\ P=\frac{W'V_o}{2ga^2}(x_1+x_2).$$

P is the mean value during the piston movement MN: but if we make MN very small, that is, make x_2 equal to x_1, P becomes the actual value at M, and its value is given by

$$P=\frac{W'V_o^2}{2ga^2}(x_1+x_1)=\frac{W'V_o^2}{ga}\cdot\frac{x_1}{a}.$$

P then varies as x; and it follows that if we draw a curve of effort by setting up, at each point M (Fig. 159), the value of P at that point, then the curve will be a straight line passing through O. For then at any point, $P = x \tan \alpha$, or P varies as x.

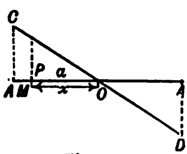


Fig. 159.

Since the curve passes through O, the part to the right of O lies below the base line, which in the ordinary way would indicate that P was negative. It can be easily seen that such is the case by considering a crank position to the right of the upright. Since x when measured to the left is taken as plus, it must when to the right be negative, and hence P becomes negative. Or, we can see that the velocity of the parts must be retarded, which requires a pull to the left. Actually,

of course, the retardation is effected by the resistance of the crank pin; and what our figure shows is, that during the latter half of the stroke the reciprocating parts can exert, independent of any steam pressure, an effort equivalent to a steam pressure on the piston represented by the ordinate of OD.

There is, of course, on the whole no total effect; for during the first half of the stroke the parts abstract from the steam energy represented by OAC, while during the latter half they give out energy represented by OA′D, and these are of course equal. Plainly this must always be the case, since we start with no velocity, and end with the same.

Correction of Indicator Diagram.—The pressure we have so far spoken of is a total pressure; but it will be convenient to express our result as a pressure per sq. in., and this we proceed to do.

Let p be the pressure per sq. inch on the piston equivalent to P. Then

$$p=\frac{\mathrm{P}}{\mathrm{A}} \quad (\text{P in lbs., A in sq. ins.}),$$

$$\therefore p=\frac{\mathrm{W}'}{\mathrm{A}} \cdot \frac{\mathrm{V}_o^2}{ga} \cdot \frac{x}{a}.$$

But W′/A is a pressure per sq. inch, being lbs. divided by sq. inches, and it is the pressure which if applied to the piston would produce a total pressure W′. Hence representing it by p_o, we call p_o the pressure equivalent to the weight of reciprocating parts. We have then

$$p=p_o \cdot \frac{\mathrm{V}_o^2}{ga} \cdot \frac{x}{a}.$$

To find its initial value put $x=a$, and then

$$p=p_o\frac{\mathrm{V}_o^2}{ga}.$$

and this pressure is required simply to start the piston.

Now the actual pressure per square inch on the piston

is given us by an indicator diagram on a certain scale of lbs. per sq. inch to the inch, thus ABCDE, BCD being the top line of the diagram taken for the particular end we are considering, and AE the bottom line of a diagram from the other end of the cylinder. Then the ordinate between BCD and AE gives us the difference between the steam pressure on the side considered, and that on the other side of the piston, *i.e.* gives the effective pressure driving the piston.

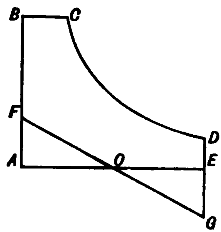


Fig. 160.

Now, on the same scale as the diagram, set up AF to represent $p_0 V_0^2/ga$ lbs. per sq. inch, bisect AE in O, and join FO, producing it to cut DE in G.

Then between A and O the pressure which is available to produce crank effort is got by subtracting the ordinate of OF from that of BCD; while between O and B there is, in addition to the ordinate of BCD, an effort produced equivalent to pressure represented by the ordinate of OG. We can represent this total effect by using FG instead of AB as a base to measure from. Then doing this the vertical ordinates between BCD and the base FG represent the effective pressures producing crank effort.

The process here given is known as correcting the diagram for inertia.

Fast Running Engines.—The figure as drawn would be for an engine running at medium speed, and it is seen that the effect is to reduce the inequality of pressure caused by an early cut-off. The effect, however, increases very rapidly as the speed increases, since $p \propto V_0^2$, and in fast running engines it becomes of great importance. In such cases F may rise to, and even above B. This latter case is shown in Fig. 161, and what it shows is that, before K, either the crank drags the piston along, the steam pressure not being sufficient

to keep it up, or else—in engines such as Brotherhood's, where the connecting-rod bearing does not encircle a gudgeon pin, but simply bears against it so that it cannot pull—the end of the rod leaves the piston.

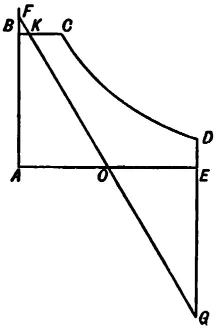


Fig. 161.

Now, directly we pass K, the piston commences to drive, thus the gudgeon pin, which was bearing against the front brass of the crosshead, Fig. 162 (*a*), dragging it, now suddenly comes into bearing with the back brass, Fig. 162 (*b*), being now driven by it. This, of course, causes a knock in the bearing, and exactly the same happens in the crank bearing. In the Brotherhood engine the piston and connecting rod end would come together, causing a knock there also.

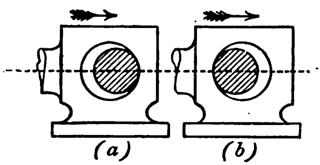


Fig. 162.

In the ordinary double-acting engine this reversal and knock must take place, and the only difference above is that it takes place at K instead of the commencement of the stroke; but in the Brotherhood single-acting engine the avoidance of reversal is one of the chief points aimed at, hence in these the speed is limited by the necessity for keeping F below B.

EXAMPLES.

1. From the diagram drawn for question 2, chap. ix., find the values of the fluctuations of energy, and the corresponding coefficients.

Ans. 22.4, 25.5, 28, 24.9 tons-ft.; .036, .042, .046, .04, coefficients.

2. An engine of 150 H. P. runs at 100 revolutions. Find the weight of a fly-wheel 10 ft. diameter to keep the fluctuation of speed within 2 % of the mean speed. Ratio of connecting rod to crank—1st, 4 to 1; 2d, 6 to 1.

Ans. 1st, $k = .1358$, whence $W = 1.76$ ton; 2d, $k = .1245$, whence $W = 1.61$ ton.

3. If there be two of the preceding cylinders on cranks at right angles, what weight of reciprocating parts would render a fly-wheel unnecessary? Stroke 3 ft.

Ans. Each set. 1st, $19\frac{1}{4}$ tons; 2d, 18 tons.

4. A vertical cylinder is supplied with steam of 50 lbs. pressure by gauge, the cut-off is at half stroke, back pressure 16 lbs., diameter of piston 40 ins., piston speed 800 ft. per minute, stroke 3′ 6″, weight of piston, etc., 2 tons. Find the effective pressure at each quarter of the up and down strokes.

Ans. Up—Commencement, 2.4; 1st, 16.4; 2d, 30.4; 3d, 27.7; end, 33.4.
Down—Commencement, 9.6; 1st, 23.6; 2d, 37.6; 3d, 34.9; end, 40.6 lbs. per sq. inch.

5. The stroke of an engine running at 250 revolutions is 8 ins., diameter of piston 8 ins., initial steam pressure 55 lbs. Find the greatest weight of piston, etc., which can be allowed, so that the connecting rod may be always in compression. The engine is single-acting. *Ans.* 388 lbs.

6. In the first case of question (2), supposing that at the commencement of the stroke the crank shaft is revolving accurately at its mean speed, find the greatest and least speeds, and thus show that the mean speed is practically their arithmetic mean.

CHAPTER XII

DYNAMOMETERS—BRAKES—AND GOVERNORS

We have fully explained in chap. iv. how the power of an engine, so far as it is shown by its Indicated Horse Power, or I. H. P., is measured. But the I. H. P. of an engine is more truly a measure of the power of the boiler than of the engine, since it gives the energy exerted by the steam, and not the work which the engine can do against the resistance. In the absence of friction, the two quantities mentioned would be equal for a whole period. But in the actual case the work done is less than the energy exerted by an amount depending on the magnitude of the friction of the machinery.

Taking now two engines of equal I. H. P., their commercial value will depend, to a great extent, on the ratio which the work done bears to the energy exerted; or, in other words, on their efficiencies; and hence the determination of the work done by an engine is of quite as much if not more importance than the determination of its I. H. P.

There are two ways in which the work done can be found, viz.—

First, by calculation of the work wasted on friction. For this purpose we should require to know accurately the laws of friction which suit the pressures, velocities, and lubricants employed; these laws are outside our present limits, and, moreover, it is doubtful if they can be given with certainty at all. Then, again, given the

required knowledge, the process would be long and cumbrous. We are thus led to consider the

Second method, an instrument is used which actually measures the work done, as an indicator does the energy exerted.

Instruments for this purpose are called **Dynamometers**, and they are of two types, the type used depending on the conditions under which the trial is to take place. If the engine is required to carry on its ordinary work while the test is being applied, a **Transmission Dynamometer** is used; while when the engine can be used for the time entirely for the purposes of the test, an **Absorption Dynamometer** may be used.

We will now examine the working of one or two examples of both these types, when the reason for their names will appear.

Transmission Dynamometer.—Fig. 163 shows one form of this type.

A is a pulley on the engine shaft, and B a pulley on the shaft to be driven, and to which the resisting moment is applied. C and D are equal pulleys, mounted on a cross-piece E, of length $2l$, which can turn round O. A drives B by means of a belt which passes under A, over C and D, and under B.

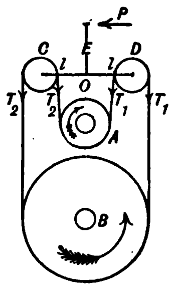


Fig. 163.

Now, when A rotates clockwise, the tension T_1 on the right-hand side becomes greater than T_2 on the left (see page 120). In passing over C and D, the tensions remain unaltered, except for the very slight friction of the axles, which we neglect. Then the difference of tension $T_1 - T_2$ causes B to turn.

The arrows in the figure show the directions of the pulls of the belt on the pulleys, and hence we see that

C is pulled down by $2T_1$, while D is pulled down by $2T_2$.

There is then a moment

$$2T_1 \times l - 2T_2 \times l,$$

tending to turn the arm clockwise round O. To balance this a spring is attached to the end of the upright arm on E, supplying a force P, the magnitude of which can be measured by noting the compression of the spring. Let y be the length of the upright arm, then we have

$$P \times y = 2T_1 l - 2T_2 l,$$

$$\therefore \; T_1 - T_2 = \frac{P \times y}{2l}.$$

But

$$(T_1 - T_2) \times \text{radius of B} = \text{resisting moment},$$

since B revolves uniformly, so that by measuring P, we find $T_1 - T_2$, and hence the resisting moment. The revolutions of B are measured separately; and hence we can at once calculate the work done per minute.

The meaning of the name can now be seen. For the energy equivalent to the work done is *transmitted* through the dynamometer, being measured during the transmission.

The form here described has been used in England by Thornycroft and Froude, and in America on a large scale by Tatham.

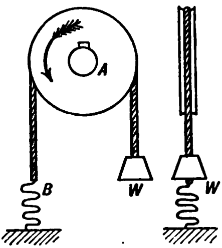


Fig. 164.

There are many other forms, but the principle is the same in all. We proceed then next to the

Absorption Dynamometer.—In this case the whole of the work done is *absorbed* in overcoming friction, hence the name.

In Fig. 164 A is a wheel with conical or plain rim fixed on the end of the engine shaft. A rope is fastened

to a spring at B, and passes round A, either once, or more turns if required. The engine turns A in the direction of the arrow, against the friction of the rope, which is prevented by the weight W from turning with the wheel.

Suppose now we wish to determine the work done by the engine when running at N revolutions per minute. Then, steam being turned on, the weight W is adjusted until it is just sufficient to keep the engine running at the required revolutions.

We have now the wheel A running at a constant speed and acted on by—

1st. The turning moment of the engine, applied to A by means of the key which connects it to the shaft, and by the friction between the shaft and the hole in A which it fits into.

2d. The normal pressures of the rope on the circumference embraced by it.

3d. The friction of the rope round the circumference.

Let us denote these actions by M, P, and F respectively. P being the sum of all the small pressures, and F the sum of all the frictions.

Then for one revolution the principle of work gives

$$M \times 2\pi = F \times 2\pi r.$$

[r being the mean radius of the wheel where the rope rests on it.]

The useful work done is zero, because each of the small normal forces of which P consists is at right angles to the motion (compare page 113). Thus there is only waste work done against F,

$$\therefore M = Fr.$$

As this is a very important point we will treat it from another point of view

$\therefore$ A is revolving uniformly.
$\therefore$ the forces acting on it are in equilibrium.

We can then take moments about any point, and the sum of the moments must be zero.

Now we wish, if possible, to exclude P from our equation; and we must therefore take our moments round O. Then P, being made up of a number of small normal forces, each of which, since it passes through O, can have no moment about O, it follows that P has no moment about O.

We have then at once, *algebraically*,

$$M + Fr = 0,$$

i.e. arithmetically, $M = Fr$, but they are in opposite directions.

[This example shows once again the importance of considering *all* the forces acting. If we omit to consider P, there is no reason for taking moments about O particularly, since the statical principle says *any* point. Thus any one, thinking only of F and M, might take moments about some other point, obtaining a different and therefore erroneous result, because round any other point than O, P has a moment; or at least may have a moment, for there are some other points than O about which P has no moment, but their determination it is not necessary to enter into.]

So far we have considered the forces on the wheel, now consider those acting on the rope. The bodies touching it are (page 97)—

	The Spring.	Wheel.	Weight.
Forces . .	Q,	P and F,	W.

Q being determined by noting the extension of the spring.

The rope is a body at rest, *i.e.* it may oscillate slightly under the small variations of the turning moment, but its mean position remains unaltered; so that the forces acting on it, taking their mean values, must be in equilibrium.

We wish still to avoid P, so we will take moments round O. We cannot now use the principle of work, since there is no motion. We have then, algebraically,

$$Fr + Qr + Wr = 0,$$

or, since Wr turns clockwise and the other two anti-clockwise, arithmetically,

$$Fr + Qr = Wr,$$
$$\therefore\ F = W - Q.$$

But we have already

$$M = Fr,$$
$$\therefore\ M = (W - Q)r,$$

W, Q and r being known by observation, hence we have determined M.

Meanwhile the revolutions are measured and are N per minute,

$\therefore$ work done against the friction per minute $= M \times 2\pi N$ ft.-lbs.

M is to be measured in lbs.-ft., *i.e.* W and Q in lbs. and r in ft.

The friction dynamometer can also be used as a brake, and hence they are often called brakes. Then the work done, when reckoned in horse power, is called the **Brake Horse Power**, in opposition to the Indicated Horse Power. We have then

$$\text{Brake horse power (or B. H. P.)} = \frac{M \times 2\pi N}{33000}.$$

This power is that which is available for doing work when we remove the dynamometer; and it is a true measure of the power of the engine as a worker.

The type we have just described is known as the Tail Rope Dynamometer; and it is simple and works very nicely, so that it is much used where the power is not very great. It has, however, a disadvantage if much friction be required, as will be the case when the power to be measured is large. The disadvantage lies in the fact that we can increase the friction F only by increasing W, and this increases the downward pull $Q + W$ on the wheel; which of course strains the shafting, and causes increased friction in the shaft bearings. The equations show this effect; for referring to them we see that the force balancing M is only the difference of W and Q,

while the force straining as above is the sum; thus it is quite possible to have a large load on the wheel, while the resisting force $W - Q$ is small, since W and Q may both be large.

Prony Brake.—To get over the above difficulty a form of dynamometer is used in which the friction force F can be increased to any extent independently of W.

A is the shaft, revolving between two blocks of wood which can be brought together by the bolts and nuts

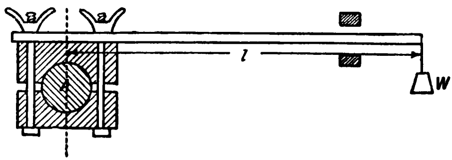


Fig. 165.

shown. Thus any amount of friction can be caused without hanging on any weight at all. The weight W, which measures the moment, is hung on the end of a long arm, so that only a small weight is required; and there are stops fitted as shown above and below the arm.

In the great majority of cases the lower block is replaced by separate blocks on an iron strap, as here shown (Fig. 166). This admits of better lubrication.

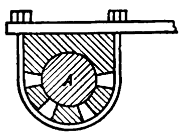


Fig. 166.

Now to measure the power at say N revolutions.

Start the engine, and screw up the nuts until the shaft revolves steadily at N revolutions; during this time the arm will bear against the upper stop.

Now adjust W till the arm just leaves the stop, hanging in equilibrium between the two. If now

l = distance of W from centre of shaft,

we have, by proceeding identically as in the preceding case,

Turning moment of engine $= Wl$.

[In comparing with the preceding take as the first body the piece of shaft inside the blocks, and consider the remaining part as an external body acting on it.]

And

$$\therefore \text{ B. H. P.} = \frac{Wl \times 2\pi N}{33000}.$$

There is one point in which the tail-rope brake has an advantage over the Prony; which is that any small discrepancies between $(W - Q)r$ and M, caused by slight variations of M, or by inequality of lubrication altering F, and hence $W - Q$, are automatically adjusted. For suppose M increases slightly, becoming momentarily greater than $(W - Q)r$; then (Fig. 164) M carries the rope round to the left; but this decreases Q, and, therefore, W being constant, increases $W - Q$; and so the balance is restored. If on the other hand M fall off, W pulls the rope round to the right, increases Q and therefore decreases $W - Q$, restoring the balance once more. Next suppose the lubrication fall off a little, then the coefficient of friction increases; so that the wheel seizes the rope, carrying it slightly to the left; this decreases Q and therefore also $W + Q$, but the total amount of friction depends on $W + Q$ the total load, so that the friction at once falls off and the rope slips back. Similarly for an increase of lubrication the balance restores itself. In the Prony brake these inequalities would keep the brake continually striking against one or other of the stops.

In actual practice then the brake is not still but oscillates continuously; but we must be careful to notice that this does not affect our results, because there is, on the whole, no velocity imparted to the brake (compare page 60), the small oscillations simply cancelling each other.

[Various means have been adopted for producing an automatic adjustment, but space prevents our examining them. If the

student should have occasion to examine such cases, let him be very careful to consider the action on the brake of *every body that touches it.* Examples of the utter confusion arising from not attending to this point may easily be found.]

Integrating Dynamometers. — Great accuracy can be obtained by making the dynamometer register the force transmitted on a paper band which is moved by connection to the engine shaft, thus a curve is traced out, the area of which gives the work done. We may look upon it as a sort of continuous indicator diagram, the principle being the same. Such dynamometers are called Integrating Dynamometers.

Brakes and Governors.—The absorption dynamometer can be used as a brake to absorb surplus energy, and in this case the problems arising are solvėd exactly as those already treated. If now we have an engine working at constant power, while the work required to be done varies, we could keep the speed constant by means of a brake, so adjusted as to absorb the surplus energy of the engine. But this would be a wasteful process, since the surplus energy would be wasted, hence for such a purpose we use, not a brake, but an instrument which cuts off partially the supply of energy, or of steam to the engine. Such an instrument is called a Governor, and we propose now to consider the action of the simplest and most general form used. Fig. 167 shows this form.

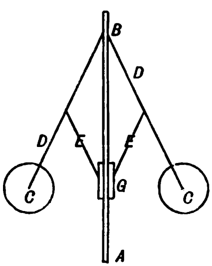


Fig. 167.

AB is a spindle rotated by the engine, so that its angular velocity bears a constant ratio to that of the shaft. CC are heavy balls connected by rods DD to a point B of the spindle, then as the spindle revolves the balls fly out and pull up the piece G, which can slide on the spindle, the pull being applied by the

rods EE. The slider G is connected by linkwork, not shown, to the regulating valve; and the proportions are so arranged that, when the engine is running at the proper speed for the work it has to do, the regulating valve is just wide enough open to enable the engine to run against the average resistance. Suppose now the resistance rise above this mean value, then the speed falls off, consequently the balls fall in towards the spindle, pushing down the slider, and the linkwork is so arranged that this opens the regulator, admitting more steam, and so enabling the engine to overcome the increased resistance. Similarly when the resistance falls off the speed increases, the balls rise, and G rises shutting off the steam.

The question we now wish to solve is—Given the speed of revolution, what position will the balls take up? and after that, supposing the speed change, what force will be exerted to move the slider?

The first of these questions cannot be solved by the use of the principle of work, there being no energy exerted or work done, consequently we must treat it by use of the laws of motion, or at least we will use the result obtained in theoretical mechanics for this case.

Circular Motion—Centrifugal Force.—The two balls move in a circle, the plane of which is at right angles to the spindle, their motion is then one case of circular motion, and hence we will first consider circular motion generally.

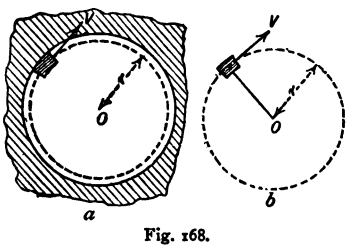


Fig. 168.

In order to cause a body to move at uniform velocity in a horizontal circle, we must do one of two things. Either we must place it

inside a cylinder of the given radius, Fig. 168 (*a*), or else we must fasten it to a centre O in the plane by a string equal in length to the given radius.

[This latter is not quite accurate, as we shall see a little further on, but is put in this way for simplicity, to keep all the bodies in one plane; also we consider a small particle, whereas in the figure we have given it definite dimensions.]

If now we start the body with a velocity V at right angles to the radius, it will, in the absence of friction, and of resistances generally, continue to move in the circle at this speed.

We can now easily see what is the nature of the actions between the body and the cylinder or string respectively.

The only action which can exist between the body and the cylinder is a direct normal pressure between their surfaces, since there is no friction. But there must be an action of some sort, because otherwise the body would go on moving with a velocity V in the same direction as we started it, *i.e.* along the tangent. Hence then there is a normal pressure, say X, between the body and the cylinder surface, or at any instant the body exerts a push X outwards on the cylinder, and the cylinder exerts a push inwards towards O on the body.

Next for the string, this plainly can exert no effect on the body other than a direct pull; by having a solid rod in place of the string (Fig. 168) and taking hold of it at O, it would be possible to apply a force at right angles to the rod, but it is evidently impossible to do any such a thing with a string, a direct pull along its length is the only possible force the string can apply. The action between body and string then is a mutual pull, the string pulls the body with a force X, and the body pulls the string with a force X also (see page 106). It appears then that a body which is constrained by contact with some other body to move uniformly in a circle, is at every instant exerting on the constraining body a force X,

outward along the radius at the instant. The force is of course constant, since all parts of the path are absolutely identical as regards motion, and therefore also force.

This force X is called the **Centrifugal Force**; we have seen that it must exist, but its value we will take as known, being determined in treatises on theoretical mechanics. If

W = weight of body,
r = radius of circle it describes,
A = angular velocity in the circle,
$= \frac{V}{r}$.

Then

$$X = \frac{WA^2r}{g} = \frac{W}{g} \cdot \frac{V^2}{r}.$$

The first is generally more convenient.

In dealing with a body of definite dimensions, r is the radius of the circle described by the C. G., and the total centrifugal force is given by putting W = weight of whole body.

We can now return to the consideration of the **Governor**.

When the whole is steadily rotating in equilibrium there will be no pull on the rods EE (Fig. 167); hence we may, in the first instance, omit them, and thus obtain Fig. 169, where also we only consider one ball, the motions being identical.

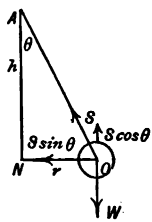


Fig. 169.

O is the centre of the ball. W its weight. Let

$$ON = r, \quad AN = h, \quad \angle NAO = \theta, \quad A = \text{angular velocity}.$$

Since the ball is rotating in a circle at a constant speed some body must be exerting on it a force X along ON, or it must be exerting on the constraining body a force X in the direction NO.

The constraining body is now the rod AO, which does not lie in the plane of the circle (see note on page 248),

but its effect on the body in that plane must be a force X along ON.

But if S be the tension of the rod, its effect along ON is S cos AON = S sin θ,

$$\therefore \mathrm{S} \sin \theta = \mathrm{X} \qquad (1).$$

Also the ball moves horizontally, and is therefore in equilibrium vertically. Whence

$$\mathrm{S} \cos \theta = \mathrm{W} \qquad (2).$$

[The rod is the only body touching the ball, therefore S is the only force acting on it besides W.]

Therefore from (1) and (2)

$$\tan \theta = \frac{\mathrm{X}}{\mathrm{W}} = \frac{\mathrm{WA}^2 r}{g \, . \, \mathrm{W}},$$

$$\therefore r \cot \theta = \frac{g}{\mathrm{A}^2}, \quad \text{and } h = r \cot \theta,$$

$$\therefore h = \frac{g}{\mathrm{A}^2}.$$

h is called the height of the *simple* governor, or the height due to the revolutions, and the result shows that it is independent of weight of ball or of the length of the rods.

The method we have used is not that usually given, nor is it a very convenient method. Usually the ball is shown, as in Fig. 170, in equilibrium under the forces S, X, and W.

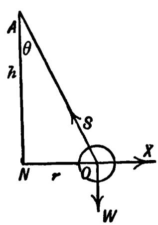


Fig. 170.

Then we have at once

$$\mathrm{S} \sin \theta = \mathrm{X}, \quad \mathrm{S} \cos \theta = \mathrm{W},$$

and hence the same results as before. But there is in this method a danger that the student may think of X as a force acting *on* the revolving body, whereas, as drawn in Fig. 170, it is the force exerted by the ball on the rod.

The forces, as drawn in Fig. 170, do not represent forces acting on the ball, but on the end of the rod

where it meets the ball. Suppose now the rod to extend to O, and to be connected to the ball by a pin passing through its centre (Fig. 171). Then if we consider the equilibrium of this pin, we have—

Bodies touching it – Rod and Ball.
Actions . . . (S – W) and X.

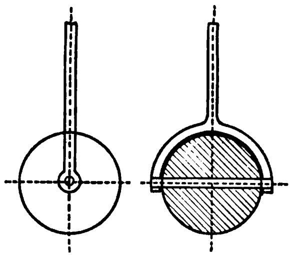


Fig. 171.

And then Fig. 170 represents correctly the forces keeping this pin in equilibrium. Results obtained then as from Fig. 170 are quite correct, always bearing in mind what we have just been saying, and hence we shall always put the forces in that form.

The next question is—If the speed change from A to A′ what pull is caused on the rods EE, and hence on the slider G? As the speed increases the balls tend to rise, but they cannot actually do so until they exert a sufficient pull on the rods EE (Fig. 167); we are now then going to consider the governor revolving at A′, but at the proper height for A, *i.e.* before the slider has moved.

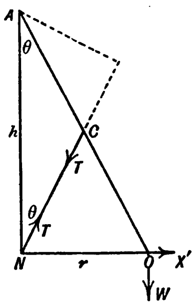


Fig. 172.

Let T be the tension of NC (we consider one only), C being the middle point of AO, and let the other forces be as usual. Then the forces acting on the rod and pin at O are T, W, X′, and the action at the point A.

To avoid considering this latter, take moments about A, and we have

$$Th \sin \theta + Wr = X'h.$$

X' is the centrifugal force due to A',

$$\therefore \ Th \sin\theta + Wr = \frac{WA'^2 r \cdot h}{g}.$$

But $g/A'^2 = h'$, the proper height for the new revolutions

$$\therefore \ Th \sin\theta = \frac{Wr}{h'} \cdot h - Wr$$
$$= Wr\frac{h-h'}{h'}.$$

Next consider the slider; it is acted on, taking now the two balls, by the two tensions T, therefore the total pull on the slider, say F, equals $2T\cos\theta$,

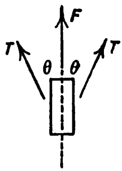


Fig. 173.

$$\therefore \ F = 2Wr\frac{h-h'}{h'}\,\frac{\cos\theta}{h\sin\theta},$$
$$= 2W \cdot \frac{h-h'}{h'}.$$

If h and h' be found, this gives at once the value of F.

Now the resistance to motion of the slider, say R, will prevent motion taking place till F equals R; so that if we know R we can find the change of speed necessary before the slider begins to move. We shall have for this

$$R = 2W\frac{\frac{g}{A^2} - \frac{g}{A'^2}}{\frac{g}{A'^2}} = 2W \cdot \frac{A'^2 - A^2}{A^2},$$

or this will give the necessary weight of the balls in order that a given change of velocity may be sufficient to move the slider. The smaller the change of speed necessary the greater is said to be the sensitiveness of the Governor.

Bursting Effect of Centrifugal Force.—This effect, mentioned on page 228, we can now explain.

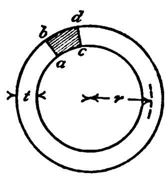


Fig. 174.

The figure represents the rim of a fly-wheel, rotating with angular velocity A. Let

r = radius of rim,
b = thickness or width perpendicular to the paper,
t = width in plane of paper.

Consider now the forces acting on 1 inch in length of the rim, contained between the two radial lines *ab*, *cd*. If

w = weight of 1 c. in. in lbs.
$w.t.b.$ = weight of 1 inch length of rim.

Then, to make this piece rotate in the circle, there must be applied to it a force

$$\frac{w.t.b.A^2r}{g}.$$

This force can only be applied by the action on the 1-inch length, shown shaded, of the remainder of the wheel; which action is applied over the plane ends *ab* and *cd*; and if A be so great that the strength of the metal is not sufficient to supply such a force, then the shaded piece will no longer travel in the circle, *i.e.* the rim will burst. For the completion of this investigation we must refer to page 274; where the nature of the actions between the small piece which we have picked out for consideration, and the rest of the rim will be better understood.

EXAMPLES.

1. In testing the power of an engine by the transmission dynamometer of Fig. 163, the revolutions of B were observed to be 300 per minute and the thrust P 10 lbs. The dimensions were, radius of B 24 ins., centres of C and D 44 ins. apart, arm E 30 ins. long. Find the brake H. P. *Ans.* 1.56

2. The power of an engine is tested by a tail-rope dynamometer. The wheel is 5 ft. diameter, the weight 300 lbs. Find the horse power when the spring balance shows a pull of 180 lbs., and the fly-wheel makes 150 revolutions per minute. *Ans.* $1\frac{4}{11}$.

3. The coefficient of axle friction is sometimes determined by hanging a heavy pendulum on a shaft so that it can revolve freely on the shaft. The shaft is then rotated at a given speed and the pendulum takes a position of equilibrium at an angle θ to

the vertical. If W = weight of pendulum, l the distance of its C. G. from the centre of the shaft, N the number of revolutions per minute, prove that the horse power lost in friction is $WlN \sin\theta/5255$. Deduce the value of f' (page 69).

Ans. $f' = \frac{l \sin\theta}{r}$, r = radius of shaft.

4. In question 2 find the effect, 1st, of a fall-off of 3 per cent in the power of the engine; 2d, of a decrease of 10 per cent in the friction due to increased lubrication.

Ans. 1st, Leaving the weight unaltered, the revolutions will drop 3 per cent, and the spring balance be unaltered; 2d, the balance will show 190 lbs., and the revolutions rise to 164 nearly.

5. In testing an engine by a Prony brake, the arm was vertical and connected to a spring balance. In order to support the weight of the brake, 720 lbs., a stiff spring was fitted on which the brake rested. This spring being supposed to be directly under the centre of the shaft; if the engine were running at 140 revolutions, show that a displacement of 1 ft. in the position of the spring will falsify the results by 19 H. P.

6. Find the height of a simple governor revolving at 75 revolutions. *Ans.* 6¼ ins.

7. The balls of a governor weigh each 3 lbs., and are each hinged to a pair of equal rods, one of each pair connected to the spindle, and the other to a slider which moves the throttle. The speed suddenly increases from 75 to 77 revolutions. Find the pull on the slider. *Ans.* .162 lbs.

8. Solve the preceding when the governor is constructed as in Fig. 167, page 246. *Ans.* .324 lbs.

9. If a plate rotate about any axis perpendicular to itself, prove that the centrifugal force is the same as if the whole mass were concentrated at the C. G., revolving with the same angular velocity.

Ans. Let P be any point, w the weight of a small particle at P, O the centre of rotation, G the C. G. The centrifugal force of w is $w/g \cdot A^2OP$ along OP. This is by statics equivalent to two forces $w/gA^2 \cdot OG$, and $w/gA^2 \cdot GP$. Adding up all these the first set gives $W/gA^2 \cdot OG$, and the second set vanish because G is the C. G. Hence the result.

10. A fly-wheel, diameter 12 ft., weighing 5 tons, is bored out

of truth, so that its C. G. is 1 in. from the centre of rotation. Find what pull this will cause on the shaft at 80 revolutions.

Ans. 2036 lbs. This pull, since it always lies along OG, will be continuously bending the shaft in different directions, and will set up very great vibration. Hence the necessity for perfect balance, especially at high speeds.

PART II

STRENGTH OF MATERIALS AND STRUCTURES

CHAPTER XIII

STRESS AND STRAIN—TENSION AND COMPRESSION

A MACHINE transmits energy by means of force, which is transmitted along the chain of pieces of which the machine consists. At each contact of a pair we have seen there is a mutual action between the surfaces in contact, which mutual action we have called a Stress (page 106).

The connections between energy exerted, work done, velocities, and the forces acting on the various pieces, have been considered in Part I. We now wish to go further, and inquire into the effects which the forces produce on the pieces to which they are applied.

For the purposes of Part I we did not require to know the transverse dimensions of the pieces; but for our present purpose these will be of primary importance.

The effects produced by forces on a piece of material depend very much on the manner in which they are applied to it; we shall consider various modes of application in order, commencing now with the case in which the forces are applied in the direction of the length of the piece. The forces applied are often called Loads even when not due to the action of gravity, and we are now about to study the effect of **Longitudinal Loads.**

The shape of the piece must be defined, and the case we consider is that of a straight rod of uniform cross-section; in most cases the section will be taken to be circular, this being most common in actual practice.

There are now two ways in which the rod may be acted on—

Fig. 175.

1st. As in Fig. 175 (*a*), there may be forces P, P acting inwards at the ends. This action is called **Compression**, and either of the forces P is the **Compressive Load.**

2nd. As in (*b*), where the forces P, P act outwards. This effect is called **Tension**, and either P is the **Tensile Load.**

In either case, either of the forces P may be the action on the end of the bar of one other piece, or the resultant action of several pieces; in any case it is called the Load on the bar.

[We notice here that the forces are taken so as to balance, the bar being supposed in equilibrium; this will be the case all through, the effect of the forces being unbalanced being not within the limits of the present book, and it is very rarely that in any case we find this effect considered.]

The load, it must be noticed, is not 2P but P, although there are two forces P. Thus in Fig. 176 the bar AB rests on the ground supporting a weight P lbs.; in this case we should naturally say the load is P lbs. But there must be an exactly equal action P on the bottom (neglecting the weight of AB itself), which we should call the reaction of the ground, or the supporting force. But we have just as much right to call the bottom force the load, and the top one the reaction; because, although we say the rod prevents the weight falling, yet we know that a perfectly accurate statement is that the rod prevents the weight and ground coming together, and we may as well say the rod keeps the ground down as that it keeps the weight

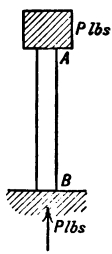


Fig. 176.

up (compare chap. i., pages 17 and 18). In either way of looking at it the load is not 2P but P.

Tension—Method of Sections.—To examine the effect of the load on the bar, we shall use, for the first time, a method of very great importance, and on the correct comprehension of which, ability to treat these questions to a great extent depends. The method is that of dividing a bar into two parts by means of an imaginary section, and then examining the mutual action between the surfaces of the two parts. For reasons which will be seen at the end of the chapter, we will take now the case of tension.

AB is a bar under a tensile load P.

Now imagine AB divided into two parts by a plane at right angles to the axis of the bar, through CD. This we call taking a transverse section through CD. Then this imaginary section divides the bar into two parts, 1 and 2. If we made an actual cut through CD, 1 and 2 would of course be separated pieces; and the idea which the student requires to thoroughly grasp is that of considering 1 and 2 as separate pieces, although no actual cut is made. This idea is not a false one, because the two pieces are actually distinct; if we had cut them and moved them 1 inch apart, they are plainly distinct pieces; now move them together till they are $\frac{1}{2}$ in. apart, they are just as distinct; and equally so when they are $\frac{1}{4}''$, $\frac{1}{8}''$, or $\frac{1}{1000}''$ apart, or if the particles of metal come into absolute contact, the fact of their touching will not make them any less distinct.

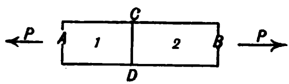


Fig. 177.

Internal Stress.—We can now find out what kind of actions must go on in the interior of the bar. For consider the piece 1 by itself. Then (Fig. 178) it is pulled to the left by P; but we know it does not go on moving to the left, but remains still. It follows

then that some force must be exerted on it to balance P.

We are not now taking gravity into account, so that the force necessary can only be applied by some other body; and the only body which touches 1, besides that, whatever it be, which exerts the load P, is 2. Whence it follows at once that the action of the surface CD of 2, on the surface CD of 1, is such as to exactly balance P.

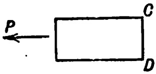


Fig. 178.

We can now consider 2 by itself, when by exactly similar reasoning to the above, we can show that the action of the surface CD of 1, on the surface CD of 2, is such as to exactly balance the other P.

We have thus proved that, between the particles of the bar on opposite sides of any transverse section, there is a mutual action consisting of two equal and opposite forces, either of which balances the load on the bar.

An action of the preceding kind we have already named a **Stress** (page 106), and we say then that the effect of the load is to produce a stress on the section CD, or on any section, CD being taken anywhere. The total amount of the stress is sufficient to balance P, and is hence equal to P. So P is the **Total Stress** on the section.

Intensity of Stress.—The word stress by itself usually means intensity of stress, or stress per unit area. In the present case, if A be the area of the section in square inches, and P be in lbs.,

$$\text{Mean intensity of stress on the section} = \frac{P}{A} \text{ lbs. per sq. in.}$$

We say the mean intensity, because, so far, we have said nothing as to the distribution of the stress over the section. If the stress be uniformly distributed then P/A is its actual value over any square inch of the section. But if the stress be not uniformly distributed, we must have some way of expressing its intensity *at a point.*

Since a point has no area, and the force on it must be also zero, we cannot find the intensity by dividing the force by the area. We have then to follow the method always adopted in expressing the value of such varying quantities, *i.e.* we estimate what would be the stress on a unit area, if it had the same intensity all over the area as it has at the point; and this amount we call the intensity at the point. At all points of a surface under uniform stress the intensity is P/A.

The present division of our work will be chiefly devoted to a determination of the distribution and consequent intensity of stress produced in a piece of material by various loadings. The question then naturally arises —Why is this quantity of importance? To which we answer, that on its value depends the ability, or otherwise, of the piece to withstand the load which is applied to it.

Strength—Limiting Stress.—There is, for every material, a certain value which the stress must not exceed; this value depending on the kind of material, and also, as we shall see in chap. xxi., on the manner of application of the load. If this stress be exceeded the piece is liable to be injured, and rendered unfit for its work. Plainly then this stress is of great importance, and it is known as the **Strength** of the material, or sometimes is called the **Limiting Stress.**

Distribution of Stress in a Stretched Bar.—We consider now the question—If a bar be exposed to a tensile load P, what is the intensity of stress produced? We shall first require to ask—How is P applied?

1st. Let P be applied with perfect uniformity over the ends of the bar.

[We do not say this is practically possible, this we shall see is unnecessary.]

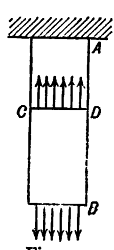


Fig. 179.

Then the part below CD is in equilibrium under a set of equal small forces all over the end B, and a set of small forces over the section CD. And

we say that in this case the distribution of stress will be uniform over CD, for there is no conceivable reason why it should be greater or less at any one point than at any other.

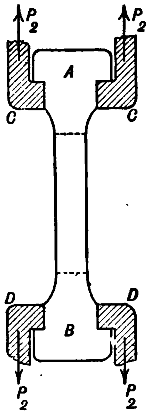


Fig. 180.

But now—

2nd. Let AB be of the shape shown in Fig. 180, which is used for test pieces, and let the load be applied by jaws CC, DD gripping AB under the collars, and which are forcibly pulled apart.

Then plainly, if we take CD very near either to A or B, the stress must be nearly all concentrated on the outer rings of the area; and it would be only by very elaborate methods, if at all, that we could determine its distribution.

In this case, however, a very important practical fact helps us, viz., although near A and B the stress is not uniform, yet the nature of materials is such that the stress very rapidly distributes itself, and at a very small distance from the ends it will have assumed a practically uniform distribution. Thus then, in the figure, the stress over any section taken between the dotted lines would be uniformly distributed.

[A principle similar to the foregoing will be found necessary in chaps. xviii. and xx.]

Line of Application of Load.—There is, however, one condition which must be satisfied before we are justified in taking the foregoing as true, this being that—*The line of action of the resultant load must be in the axis of the bar.* By axis we mean a line passing through the centres of gravity of all the transverse sections.

We have now obtained the following result:—

If a tensile load P be applied in any manner to the end of a bar, provided only that the resultant action

passes through the C. G. of the sections, then for any transverse section except very near the ends we have

$$\text{Intensity of stress at any point} = \frac{P}{A} \text{ lbs. per sq. inch.}$$

This we generally denote by p, and thus we have $p = P/A$.

We have seen that p must not exceed a certain value, viz. the strength of the material; for this value we shall use the symbol f, then the greatest load which the bar can safely bear will be given by

$$f = \frac{W}{A}, \quad \text{or } W = fA.$$

Extension under the Load.—We have next a different effect to consider, *i.e.* the change of length of the bar which is produced by the loading.

In order to find what this will be we must go to experiment, subjecting bars of various sectional areas and lengths to given loads, and noting the extensions produced. The effects obtained will be fully discussed in chap. xxi.; for the present we will consider only that one which is perhaps the most important of all, and which we need for our present purposes. This one is as follows:—

Hooke's Law—Strain.—Let

l = length of bar,
A = sectional area,
P = load,
x = extension under the load.

Then, *so long as* P/A *does not exceed a certain limit*, x varies directly as P for the same bar, or x/l varies as P/A for different bars of the same material.

Now $P/A = p$, the intensity of stress, or briefly the stress. And

$$\frac{x}{l} = \frac{\text{total extension}}{\text{length}},$$
$$= \text{extension per unit length.}$$

To this latter quantity we apply the name **Strain**, or **Intensity of Strain**. The meaning which we attach to strain, then, will be different from that in which it is often used in common language, when a strain of so many tons is often spoken of. For this we use the term stress or load, keeping strain to denote alteration of shape.

The experimental fact above can then be briefly stated thus :—

Strain varies directly as the stress.

This is generally spoken of as Hooke's Law, after the name of the original discoverer.

The strain per unit length is generally denoted by the letter e, and we have then

$$e \propto p.$$

e is simply the ratio of x to l, so that it is not dependent on the units of length employed, but is a simple number. In order to find the value of e for a given p or *vice versâ*, it is necessary to have an equation connecting them, and this equation is, since $p \propto e$,

$$p = \mathrm{E}e,$$

E being a constant, the value of which must, for any given material, be determined from experiment, and which is called the **Modulus of Elasticity**.

We can from the preceding equation give a definition, or perhaps it would be better to say a means of remembering what E is. For suppose $e = 1$, that is, the extension equals the original length, then $p = \mathrm{E}$, whence it appears that E is the stress which would double the length of a bar, *if Hooke's Law held good for such an extension.*

We have italicised the last words, because, as we shall see in chap. xxi., the law does not hold good unless the extension is very small. This shows, however, that E must be stated in the same units as p, and its value will thus depend on the units in which p is

stated. For wrought-iron or mild steel E is 29,000,000 in lbs. per sq. inch, or 13,000 in tons per sq. inch. Values for other materials are given in chap. xxi.

Limits of Elasticity.—On page 265 we have italicised the words "so long as P/A does not exceed a certain limit"; it is of great importance to bear this carefully in mind. This limit is, for reasons explained in chap. xxi., called the Elastic Limit, and so long as P/A or p does not exceed this, the material is said to be in the elastic state.

In any case Hooke's Law holds good only within the limit stated, and the whole of our work in the present and subsequent chapters which will depend on this law will be subject to the same limitation.

It must not be imagined that this detracts from the usefulness of the work, because, as we shall see, we are obliged in all practical cases, for very sufficient reasons, to keep the stress within this limit. For wrought-iron this limit is about 24,000 lbs. per sq. inch, and in all practical cases the stress on a wrought-iron bar is very far below this, being usually not more than 4 tons, or 9000 lbs. about, per sq. in. For steel and other metals the limit differs; its value is given for some common materials in chap. xxi.

We have now means by which we can find the stress and strain produced in a bar of given dimensions by a given load. For example, a bar 2 ins. square is loaded with 10 tons; material, wrought-iron. Find the stress and strain produced. Then

$$P = 10 \text{ tons}, \quad A = 4 \text{ sq. ins.},$$

$$\therefore p = \frac{P}{A} = \frac{10}{4} = 2.5 \text{ tons per sq. in.}$$

Also

$$p = Ee, \quad \text{or } e = \frac{p}{E}.$$

The material being wrought-iron,

$$E = 13{,}000 \text{ tons},$$

$$\therefore e = \frac{2.5}{13{,}000} = .0002 \text{ nearly,}$$

showing how very small e is in all practical cases.

We have found the strain e without requiring to consider what length the bar is, and the meaning is that every inch original length stretches to 1.0002 ins., or every foot to 1.0002 feet. Suppose then we are now given in addition that the bar is 18 ft. long, then it will stretch to

$$18 \times 1.0002 \text{ feet,}$$

or the total extension is

$$\begin{aligned} 18 \times .0002 &= .0036 \text{ ft.,} \\ &= .0432 \text{ ins.,} \end{aligned}$$

or not quite $\frac{1}{20}$ inch.

Bar of Varying Section.—Strictly speaking, the result $p = P/A$ applies only to bars of uniform cross-section, but it can also be used in cases where the cross-section varies, so long as the variation is a gradual one. If the change of section be abrupt, then the formula does not hold for parts very near the point of change. For example, ABC is a rod from which P hangs, then in the case (a) the stress very near B on either side could not be found by dividing P by the area of the section, because the distribution of stress near B would be unequal, and this would be shown practically by the fact that under a heavy load the bar would in all probability break at B. For this reason such a bar would not in practice be cut square in, but the corner would be rounded as in (b). Even in case (b) there

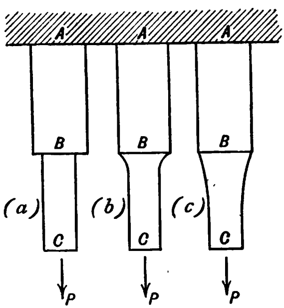


Fig. 181.

would be a little doubt as to the application of the formula to sections in the round, but when we come to cases such as (*c*) the formula may be used with perfect safety.

We cannot, however, calculate the extension of such a bar with the mathematical knowledge with which the student is at present credited, because since p varies from point to point, e, or the rate of extension, also varies, and to find the extension of a given length requires the use of the Calculus. When, however, a bar has definite lengths of different diameters, but the diameter constant for each length, then we can find the total extension by finding that of each length separately and adding the results.

Work done in Stretching.—When a bar is stretched, the stretching body, *e.g.* the jaws of the testing machine, moves against the resistance of the extending bar; hence work is done, and it is of importance to know how much. Let the bar AB be held at A, and stretched steadily by applying a gradually increasing load or pull to B. Suppose it be thus stretched to the length AC, BC being in the figure greatly exaggerated for clearness. Then, since the stretching is gradual, and BC is such a very small length (see preceding example), none of the parts will have any measurable velocity, or K. E., consequently we have at each instant balanced forces, the external load and the resistance of the bar. If then R be the final load, R is also the final resistance of the bar.

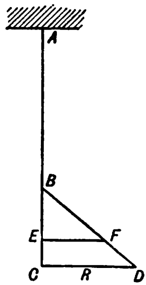


Fig. 182.

In order now to calculate the work done we must know how the resistance varies. This is very easily seen.

For at any point E of the path BC of the moving body, we have, by Hooke's Law,

Resistance at E : R = BE : BC.

If then we set off at C, CD to represent R graphically; and at E, EF to represent the value of the resistance at E,

$$EF : CD = BE : BC.$$

Therefore F lies on the straight line joining D and B, and this line is the Curve of Resistance.

$$\therefore \text{Work done} = \text{area of BCD},$$
$$= \tfrac{1}{2}BC \times CD,$$
$$= \tfrac{1}{2}x \times R,$$

x being the extension BC.

The mean resistance is thus $R/2$. The energy exerted by the effort is in this case also $Rx/2$. But it must be noticed that the above calculation of *work done* holds, even if there be not at every instant a balance between the stretching force and the resistance. Hooke's Law is properly a relation not between load and extension but between resistance and extension, for it is verified by applying loads and measuring extensions, the bar being in each case in a state of equilibrium, so that resistance = load, and, therefore,

$$\text{Load} \propto \text{extension}$$

is equivalent to

$$\text{Resistance} \propto \text{extension}.$$

Now, we assume as a fundamental axiom that the resistance to extension depends on the amount of stretch and not on the particular load which may be hanging on the end of the bar. Thus if a certain bar supports a weight of 1 ton, and is then stretched say $\frac{1}{100}$ inch, the ton hanging quietly; then we say that whenever that bar is stretched $\frac{1}{100}$ inch, its resistance will be 1 ton, even although there may be 2 or 3 or any number of tons connected to its end.

For what happens in this latter case we refer to chap. xxi., page 423, but it is well to impress on the student at the commencement that stress depends on strain,

and not necessarily on the external load producing the strain, except for the case of balance, *e.g.* to prove $p = P/A$ we were obliged to consider the whole bar and each of its parts to be in equilibrium.

Resilience.—The work done in stretching can be expressed in terms of the stress produced and the volume of the bar. For let

l = length (inches), A = sectional area (sq. ins.),
p = final stress produced, x = extension,

then

$$p = \frac{R}{A}, \quad \therefore R = pA,$$

and

$$p = E\frac{x}{l}, \quad \therefore x = \frac{pl}{E},$$

$$\therefore \text{Work done} = \frac{Rx}{2} = \frac{p^2Al}{2E}.$$

But Al = volume of bar,

$$\therefore \text{Work done} = \frac{p^2}{2E} \times \text{volume}.$$

Let now the bar be stretched till the proof stress f_p is reached, then

$$\text{Work done} = \frac{f_p^2}{2E} \times \text{volume}.$$

This quantity of work is of importance, since it represents the greatest amount of work which can be done on the bar, or of energy which can be stored in it. For the bar would on again extending exert this amount of energy, subject, however, in all actual materials to a certain loss, due to what may be called internal friction, and which appears as heat, without overpassing the limit of elasticity. It is hence denoted by a special name, viz. Resilience. Hence

$$\text{Resilience of a stretched bar} = \frac{f_p^2}{2E} \text{ volume}.$$

$f_p^2/2E$ is called the Modulus of Resilience.

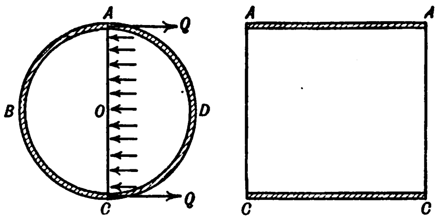
Stress in a Thin Cylindrical Shell.—The stress

we are now about to investigate does not appear at first to be a case of simple tension, nor, to be strictly accurate, is it really so, but for all practical cases it may be taken to be so.

ABCD represents a section through a cylindrical boiler, O being the centre; and the boiler is exposed to internal steam pressure. Let

$$\left.\begin{array}{l} l = \text{length of boiler} \\ D = \text{diameter} \\ t = \text{thickness of plates} \\ p = \text{pressure by gauge} \end{array}\right\} \text{inches and lbs.}$$

It is immaterial whether by D we mean the internal or mean diameter, since t is small compared to D.



(*a*) Fig. 183. (*b*)

The boiler shell is subjected to a pressure $p + 15$ lbs. per square inch inside, and 15 lbs. per square inch outside, so the effect is the same (practically) as if the intensity of pressure inside were p and outside zero.

Take now a longitudinal section by a plane AC through O; and consider the equilibrium of a body composed of the half shell ABC *and its contents*—steam or water or both. This body is acted on by a pressure p all over the surface AACC in Fig. 183 (*b*), which shows an elevation of the longitudinal section, this pressure being applied by the surface AACC of the steam and water in the other half ADC. To balance this there is only the stress on the section of the plates, and thus we see that there must be a tension in the plates supplying

forces Q, Q. To supply these forces we have the longitudinal sections of the plates of the cylindrical portion or the **shell plates**, and also the cross-sections of the two ends. We always, however, neglect the effect of the ends, and this will cause our results to be decidedly untrue near the ends; but if we consider a ring of plate, say a foot long, in the centre of a boiler 12 or 13 feet long, then this ring obtains very little support from the ends, and our results will be then practically correct. In any case it must be noticed they err on the safe side.

Leaving out the ends then we have—

$$\text{Sectional area of plate} = 2(l \times t),$$

therefore if q be the tensile stress,

$$\text{Resistance of plate} = 2qtl.$$

This balances a pressure p on an area $l \times \text{D}$,

$$\therefore\ 2qtl = p \times l\text{D},$$

$$\therefore\ q = \frac{p\text{D}}{2t}.$$

Thus we find q for a given boiler; or if, as is most common, we are given f, the strength allowed, then

$$f = \frac{p\text{D}}{2t},\quad \therefore\ t = \frac{p\text{D}}{2f},$$

which gives the necessary thickness of plate.

It must be noticed with regard to the value of f that a boiler being composed of plates riveted together, the strength f must not be taken as that of the solid metal, but only of the metal in the joint. Now the joint is always less strong than the solid, the ratio being called the efficiency of the joint (chap. xx. page 395), so that

$$\text{Strength of joint} = \text{efficiency of joint} \times \text{strength of solid metal}.$$

If the f be taken, as it usually is, to denote the strength of the solid metal, the equation for t becomes

$$t = \frac{p\text{D}}{2f \times \text{efficiency of joint}}.$$

T

The joint here will be that connecting together the separate plates of a ring, *i.e.* the longitudinal joints of the boiler.

Stress on a Transverse Section.—The pressure on the ends of a boiler tends to separate them, and thus produces a tensile stress on a cross-section of the boiler. This is, however, of little importance, because in actual boilers having flat ends these ends are connected by stays which relieve the shell plates from any tension in the longitudinal direction. There is one case in which this is not done, which we will set as an example.

Bursting Stress due to Centrifugal Force.—We can now complete the investigation on page 252. For we have there seen that the sections *ab*, *cd* (Fig. 174) must apply to the piece *abcd*, an action sufficient to have a resultant $wtbA^2r/g$ passing through the centre. But this is exactly what these sections would have to do if there were an internal pressure wbA^2r/g on the rim, and it follows therefore that there must be a tension q on these surfaces given by

$$q=\frac{wA^2r^2}{g}=\frac{wV^2}{g}.$$

This equation then gives the tension caused by the centrifugal force, or perhaps it is better to say—this is the tension necessary to keep the piece revolving in the circle.

Compression.—We leave compression to the last because there is a difficulty connected with it which does not appear in tension. When a rod is stretched it remains straight if originally so, whether it be 1 inch thick or $\frac{1}{10}$ inch thick. But when a thin bar is compressed it always has a tendency to bend sideways, and hence experiments are difficult to carry out except on very short pieces. If we have long pieces we must put them in a sort of trough to prevent this bending, and then the sides of the trough may act on the piece in other ways, and our results are not so trustworthy.

Generally we may say, within the elastic limit, there is little difference between the laws of tension and compression. The values of E and the proof stress are about the same (see chap. xxi.), and the calculations of stress and work done will be the same, if the bar be such as to be in simple compression. The strength of columns of ordinary dimensions, however, can only be properly determined by a formula derived from special experiments, and which is given in chap. xxi.

EXAMPLES.

1. Find the stress produced in a pump rod 4 ins. diameter, lifting a bucket 28 ins. diameter; the pressure on top of the bucket being 6 lbs. per square inch in addition to the atmosphere, and a vacuum below of 26 ins. by gauge; taking each inch as $\frac{1}{2}$ lb. *Ans.* 925 lbs. per square inch.

2. The rod in the preceding is 5 ft. long, find its extension. Material of rod brass, for which $E=9,000,000$. *Ans.* .0061 ins.

3. Assuming the strength of a chain to be double that of the bar from which the links are made, find the proper size of chain for a 20-ton crane, using three sheaved blocks; allowing $f=6000$ lbs. *Ans.* 1 inch.

4. A steel piston rod is 8 ins. diameter; the diameter of cylinder being 88 ins., and effective pressure 40 lbs. per square inch. Find the stress produced, and the total alteration of length during a revolution. Length of rod 9 ft. $E=29,000,000$. *Ans.* 4840 lbs. per square inch, .036 ins.

5. A cylindrical boiler, 12 ft. diameter, is constructed of $\frac{38}{32}''$ steel plate. The test pressure applied is 245 lbs. per square inch. Find the stress produced in the solid metal, and hence deduce the stress in the metal of the joints, the sectional area being there reduced to .77 of the solid. Find also the increase of diameter under the test, neglecting the joints. *Ans.* 19,500, 25,300 lbs. per square inch, .097 ins.

6. A cylindrical vessel 6 ft. diameter, with hemispherical ends, is exposed to internal pressure of 200 lbs. per square inch above the atmosphere. It is constructed of solid steel rings riveted together. Find the necessary thickness of metal, taking $f=7$ tons per square inch. Also find the longitudinal stress in

the metal of the ring joints, the sectional area being reduced to $\frac{7}{10}$ of the solid plate. *Ans.* $\frac{1}{2}$ in., 10 tons.

7. In question 4 find the work done in extending or compressing the rod; and also find the resilience of the rod. $f = 12$ tons. *Ans.* 2, 60 inch-tons.

8. In question 5 the flat ends of the boiler are stayed by steel bar stays, pitched 16 inches apart, both vertically and horizontally. Find the necessary diameter of stay that the stress per square inch at the test pressure may not exceed 18,000 lbs. per square inch. *Ans.* $2\frac{1}{8}$ ins.

9. Find the area of the base of a stone column carrying a load of 5 tons, allowing a crushing stress of 150 lbs. per square inch. *Ans.* 75 square ins.

10. An iron rod is suspended by one end. Draw a curve showing the stress at any section, and find the length of a rod which can just carry its own weight, allowing $f = 9000$ lbs. per square inch. *Ans.* Curve is a straight line; 2700 ft.

11. An iron bar 18 feet long, $1\frac{1}{2}$ in. diameter, is heated to 400° F.; nuts on its ends are then screwed up so as to bear against the walls of a house which have fallen away from the perpendicular. Find the pull on the walls when the bar has cooled to 300° F. Coefficient of expansion of iron .0000068 per degree Fahrenheit. *Ans.* 35,000 lbs.

12. A bar of iron is placed between two bars of copper of the same section and length, and the ends are rigidly connected together when at a temperature of 60° F. Find the stresses in the bars when the temperature is raised to 200° F. Coefficient of expansion of copper .0000095, $E = 17{,}000{,}000$. *Ans.* Iron, 5920; copper, 2960 lbs. per square inch.

13. A bar of 2-inch round iron, 3 ft. long, is turned down to 1 inch diameter over the centre foot length. Compare its resilience—1st, with that of the original bar; 2d, with that of a uniform bar of the same weight. *Ans.* 1 : 8, 1 : 6.

CHAPTER XIV

TRANSVERSE LOADS—BENDING AND SHEARING

WHEN the forces acting on a bar of material act, not along the axis, but transversely to it, the bar is called a **Beam.** In most cases the forces all act in or parallel to one plane passing through the axis of the bar, and we shall confine our work to this case.

Our problem then is—To investigate the straining actions in a beam, subjected to forces all in one plane.

The term straining action here used means any action tending to strain—*i.e.* alter the shape of—the piece of material. Thus tension and compression already considered are two cases, and the simplest, of straining actions.

AB is a beam, which for definiteness we will suppose to be fixed at A to a wall, and loaded with a weight W at B. Any other case might be taken, as what we are going to say will apply generally. Take now a transverse section of the beam at any point as KK. Then the body BK is in equilibrium under the action of W and of the forces which the end KK of AK exerts on the end KK of BK. These forces are the stresses on the section KK; and we see that the action of the load W is to produce stresses which can just balance the effect of W, and so keep the piece BK in equilibrium.

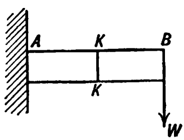


Fig. 184.

We first then inquire, What is the effect of W? For this purpose imagine the beam actually cut through at KK; then BK will fall to the position shown in (*a*). Now there is here a double effect, viz. falling, keeping parallel to itself, and turning round K; each of these is shown separately, the first in (*c*) and the second in (*b*). The combination of the effects (*b*) and (*c*) would evidently produce (*a*).

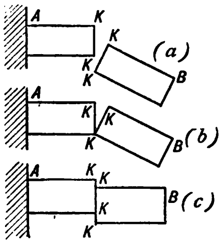


Fig. 185.

For reasons which will appear plainly as our work progresses in the present and succeeding chapters, it is convenient to take these effects separately; and we will commence then with effect (*b*).

Bending Moment.—The effect (*b*) may be expressed as a turning or bending round K as a joint; and it would occur even if the two parts AK, BK were not separated but united by a pin joint. In this case effect (*c*) could not take place. Now the magnitude of the actions or stresses on the end KK of BK must be such as to enable them to prevent this turning; and hence we must have

Moment of stresses resisting bending = moment of W about K (page 64),

W being the effort and the stresses the resistances.

We are not now going to inquire into the magnitude of these stresses, but since they are due to bending, and their magnitude depends on the *moment* of the load W, we call this moment the **Bending Moment** at KK.

We have taken for simplicity one load W; but all we have said will apply equally well, no matter how many forces we apply to BK; some may act upwards, and others downwards. In all cases we have BK in equilibrium, under the actions of all the loads acting on it, and of the stresses on the end KK, and hence

Moment of internal stresses = Sum of the moments of all the external forces acting on BK about K.

Sum here means of course algebraic sum, allowing for sign, *e.g.* in Fig. 186 the sum is, counting clockwise turning plus,

W.BK − P.CK + Q.DK − R.EK,

and this is the bending moment at K.

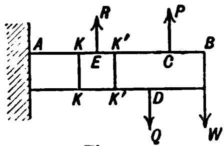


Fig. 186.

Since now KK is any section, we can define, for any section, the term bending moment as the moment about the section of all the forces on the *right* of the section. For example, for the section through K′ (Fig. 186), the moments about K′ of W, P, and Q only would be taken, since R is not a force affecting the equilibrium of K′B, which would now be the piece considered.

[There may appear to be an ambiguity in writing of the moment of a force about KK, since strictly we can only have moments about a point; but all points in KK are equidistant from W, KK and W being parallel, so the moment about KK is really about any point in KK.]

Shearing Force.—Let us now return to the effect (*c*), Fig. 185. This is a bodily vertical movement, and to prevent it the end KK of AK must exert on KK of BK forces or stresses sufficient to keep BK in equilibrium vertically. The total amount of this stress must therefore be, in Fig. 185, a vertical force W, in Fig. 186, a vertical force W − P + Q − R; then the forces balance vertically, and BK is in vertical equilibrium. The magnitude then, generally, of this action is the algebraic sum of the forces acting on BK, or to the *right* of the section; and, since the effect produced is that which would be caused by the jaws of a shearing machine, this sum is called the **Shearing Force** at the section.

Both with bending moment and shearing force we have spoken of the forces on the *right* of the section

only, but there is no particular property attached to the right side more than to the left, and hence a very probable conclusion to draw is, that on the *right* may mean simply on one side only, *either right or left but not both.* This, we shall see, is correct.

For take the beam AB as a whole, acted on by any number of forces either up or down—fixed, if we please, to the wall at A, but then this only means that the wall applies some of the forces, and so this case is included in the preceding statement—and take KK any section.

Then AB is a body in equilibrium.

[When the forces are first applied AB bends, and while bending is not a body in equilibrium strictly; but the bending goes so far and then stops, and our work all applies after this balance is attained. Compare page 270.]

Therefore the sum of the resolved parts of the forces in any direction is zero; and the sum of their moments about any point is zero.

Take then K as the point, and divide the forces into two sets, to the right and left of K respectively. Then we have

$$\text{Algebraic sum of forces} = 0,$$

$$\therefore \begin{array}{c}\text{Algebraic sum of forces}\\ \text{to left of K}\end{array} + \begin{array}{c}\text{Algebraic sum of forces}\\ \text{to right of K}\end{array} = 0,$$

and hence we see that we shall get the same numerical value for the shearing force if we take the forces to the left as if we took those to the right; there is, however, an apparent difference in sign, for if one of the above terms be plus the other is necessarily minus.

Exactly similar work applies to the bending moment, and there will be the same apparent difference as to sign.

This difference in sign is, however, not real but only apparent, and we should expect to find it. For, as we have seen, a stress consists of equal and opposite forces, and if the set of forces on the right give us one part of the stress, those on the left will naturally give us the

other part. The question as to which of the results plus or minus to use we will consider in an actual example, in any case we can give now the following

Definitions.—The shearing force, or S. F., at any point, or on any transverse section, of a loaded beam is the algebraic sum of all the forces acting on either side of the point or section.

The bending moment, or B. M., at any point of a loaded beam, is the algebraic sum of the moments of all the forces on either side of the point about the point.

[Particular attention must be paid to the words *all* and *either*. It is plain that no correct result can be obtained if some of the forces acting be omitted, and for this purpose every body touching the beam must be credited with exerting a force on it without it be expressly stated or proved not to do so. Then as regards *either*, the whole of the preceding work is based on the consideration of only one set of forces, and hence if forces from the other set are brought in the whole work falls to the ground.

Some little confusion is sometimes caused by this last, as it looks as if we were neglecting the effect of one of the sets of forces. This is, however, a false idea. Both sets together produce the bending action, and its amount is measured by either. Compare tension, page 262.]

We will now consider some examples.

Case I—Beam supported at the Ends, loaded at some Intermediate Point.—The beam rests on supports at A and B, and a load W rests on it at C. Take any point K, then we require to find the B. M. and S. F. at K. These quantities we shall always denote by the symbols M_K and F_K respectively.

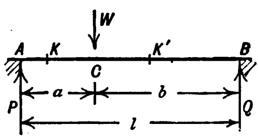


Fig. 187.

First: Find *all* the forces acting, at least on one side of K, but we shall find it better generally to find those on both.

The bodies touching the beam are three, viz. the load and two supports. We have to find the forces exerted by the latter.

Let these be P at A and Q at B, represented by the arrows.

[It is a gain in clearness and sometimes an aid to the avoidance of error, if the arrows are put on the same side of the body acted on as the body exerting the force is. Thus P and Q we put under the beam, because the supporting bodies are under. If the ends were supported by ropes, we should put P and Q above (see Fig. 188).]

Fig. 188.

Let also

$$AB = l, \quad AC = a, \quad BC = b,$$

then, since AB is in equilibrium under P, Q, and W, we have, by taking moments about B,

$$P \times AB = W \times BC,$$

$$\therefore \; P = \frac{Wb}{a+b}, \quad \text{or} \quad \frac{Wb}{l}.$$

Similarly

$$Q = \frac{Wa}{a+b}, \quad \text{or} \quad \frac{Wa}{l},$$

by taking moments about A. Or we can obtain the same result from

$$P + Q = W.$$

Now to the left of K there is only the one force P.

$$\therefore \; M_K = P \, . \, AK,$$
$$F_K = P,$$

and this result holds for all positions of K in which P only is on the left of it, *i.e.* for all sections or points in AC. It follows that

$$M_C = P \, . \, AC = \frac{Wab}{a+b},$$
$$F_C = P.$$

But if K be taken to the right of C as K′, we have on its left both W and P, so that

$$F_{K'} = W - P.$$

It is, however, now simpler to consider the forces to the right of K′, since there we have only Q, whence

$$F_{K'}=Q,$$
$$M_{K'}=Q.BK',$$

evidently the two values of $F_{K'}$ are the same, since $W-P=Q$.

And we can easily see that the same value is obtained for M either way. For looking to the left

$$\begin{aligned} M_{K'} &= P.AK'-W.CK', \\ &= P.AK'-(P+Q)CK', \\ &= P.AC-Q.CK'. \end{aligned}$$

But $P.AC$ or $Pa=Qb$, by taking moments about C.

$$\begin{aligned} \therefore\ M_{K'} &= Q.b-Q.CK', \\ &= Q.BK', \end{aligned}$$

the same result as before.

These values apply to all points in BC, hence at C we have

$$F_C=Q,$$
$$M_C=Q.BC=\frac{Wab}{a+b}.$$

Signs of B. M. and S. F.—The B. M. and S. F. at C have now been obtained in two ways—1st, by taking C as a point in AC; 2d, by taking C as a point in BC.

Taking B. M. first, we have obtained in each way

$$M_C=\frac{Wab}{a+b}.$$

But although the same result is obtained numerically, yet that obtained first gives M_C a clockwise moment, while the second gives it anti-clockwise. But, as we have before said, this is not a real difference, but depends entirely on the point of view. Looking at the way the bar would bend if jointed at C (Fig. 189),

Fig. 189.

there are three ways of describing it. Supposing an observer fastened to AC, he would say BC turns left-handed; while one fastened to BC would say AC turns

right-handed. The full statement, however, would be that of an observer standing on the ground, who would say AB bends downwards; so that an anti-clockwise moment on the right, and a clockwise one on the left are both bending of the same sign.

There is, however, a bending which is plainly of an opposite kind to that which we have just considered, viz. as Fig. 190. Here the ends bend downwards, and the moments required to produce this are clockwise on the right and necessarily, therefore, anti-clockwise on the left. These bendings then require to be distinguished by difference of sign, so that if the first or centre drooping be plus, the second or ends drooping is minus.

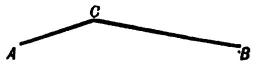


Fig. 190.

There is no necessity to specify one particularly as plus always; so long as in each example it is clearly stated which is to be considered plus, and the proper sign is kept to all through that example. The first kind being the most common, we shall, when it is necessary to discriminate, use the plus sign for it as here shown.

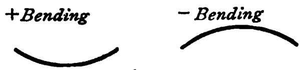


Fig. 191.

With regard to F_C there are two difficulties—first, sign, and second, there are two different values. We will take first the question of sign, and for this we need not consider C especially, other points will do rather better.

Take, for example, K. Then, looking to the left,

$$F_K = +P,$$

if we call upward forces plus. But to the right,

$$F_K = +Q - W = -P,$$

so here again are the plus and minus values.

The explanation is exactly as for bending: $+P$ on the right and $-P$ on the left constitute only one action;

and if the beam be cut, plainly "BC rises relative to AC" is just as true a description of what occurs as "AC falls relative to BC."

But look now at the point K′. Then, to the left,

$$F_{K'} = +P - W = -Q.$$

But to the right

$$F_{K'} = +Q.$$

Here then is an exactly opposite kind of action: − on right, + on left; or we may say right hand falls, left hand rises, *i.e.* relative to each other.

These two then constitute opposite or plus and minus shears, and we will take for definiteness the signs as shown in Fig. 192.

Fig. 192.

Double value of S. F.—We have found two values for F_C,

$$-Q, \quad \text{or } +P,$$

and we have to explain the reason of this.

The explanation is, that we have taken W as applied *at a point* C. Now this is a physical impossibility, for W must cover some definite distance, however small.

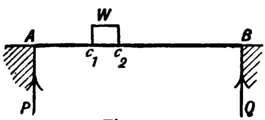


Fig. 193.

Let then W cover the small distance C_1C_2, then

$$F_{C_1} = +P,$$
$$F_{C_2} = -Q,$$

and between these it is intermediate. In taking C as one point we confuse the essentially different points C_1 and C_2. Returning now to the usual statement, considering C as one point, we should say that—In AC, F is + P; in BC, F is − Q; while at C it changes from + P to − Q. Remembering that "at C" really means between C_1 and C_2.

Diagrams of B. M. and S. F.—The graphical method is very suitable for recording the values of B. M.

and S. F., by drawing in the usual manner curves to represent those quantities.

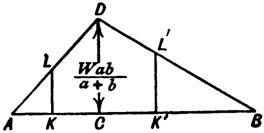


Fig 194.

In the present case we have :—

$M_{K'} = Q \, . \, BK'$ (see Fig. 187).

Set up then K′L′ on a scale of moment, *i.e.* lbs.-feet to the inch, to represent $M_{K'}$, *e.g.* if $Q = 20$ lbs., $BK' = 2$ ft.

Then

$$M_{K'} = 40 \text{ lbs.-feet.}$$

Then take say a scale of 50 lbs.-feet to 1 inch, on which $M_{K'}$ will accordingly be represented by $\frac{4}{5}$ inch. We should then make

$$K'L' = \tfrac{4}{5} \text{ inch.}$$

Doing this for all points, and drawing a curve through the points L′ so found, we should have a curve of B. M., the ordinate of which at any point would give us the B. M. at that point.

We can, however, see what this curve will be without actual construction.

For Q being constant, $M_{K'}$ or K′L′ is proportional to BK′. So that L′ lies on a straight line through B.

This holds true for all points between B and C. But between A and C

$$M_K = P \, . \, AK,$$

i.e. M_K is proportional to AK. So the curve of B. M. here is a line through A.

We have then the curve, as in Fig. 194, consisting of two lines BD, AD, meeting over C, because M_C is the same from whichever side we start.

To draw this curve then we only require to know the height CD or M_C. But this we already have, viz. $Wab/(a+b)$ (page 283).

The curve of B. M. then is a triangle of height $Wab/(a+b)$, and having its vertex directly over W or C.

To verify draw KL.

Then the moment represented by KL is

$$\frac{KL}{CD} \times \frac{Wab}{a+b} = \frac{AK}{AC} \times Pa = P \cdot AK,$$
$$= M_K.$$

Hence the curve gives the value of the B. M. Next for the S. F.

This is very simple in the present case. For at any point K in AC, F is $+P$. Set up therefore on a scale of lbs. to the inch AE to represent P, and draw EF parallel to AC.

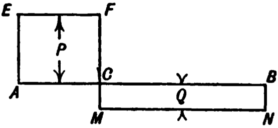


Fig. 195.

Then EF is the curve of S. F. for AC, because its ordinate at any point is $+P$.

In BC, F at any point is $-Q$. Therefore set downward BN to represent Q on the given scale, and being below the base shows the $-$value. Draw NM parallel to BC, then NM is the curve of S. F. for BC.

Draw now MCF through C at right angles to AB, and this represents the change at C from $+P$ to $-Q$.

The whole curve of S. F. is then EFCMN.

S. F. at Ends.—The S. F. at A is by the diagram $+P$ and at B $-Q$; but A and B are the ends of the beam, and there can be no S. F. or B. M. or any other action on the free ends, such as A and B, of a beam. There is here then an apparent discrepancy, which is explained exactly as the double value at C. There can be no supporting force *at a point* A, but only over some distance; and in this distance the S. F. changes from o to $+P$. Similarly for B.

Particular Case.—When the point C is the centre, then we have the greatest B. M. at C, or

$$M_{max.} = M_C = \frac{W \cdot \frac{l}{2} \cdot \frac{l}{2}}{l} = \frac{1}{4}Wl.$$

And the S. F. is $\pm W/2$.

Case II—A Lever loaded at the Ends, and Turning in a Fulcrum between.—There is now no need to examine this case in detail, because it is only Case I turned upside down. P and Q are reversed, and P + Q, which must act at C, is simply W of the first case also reversed. The diagrams of Case I then need only turning upside down, thus

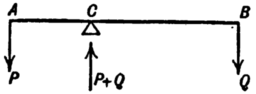


Fig. 196.

B. M.

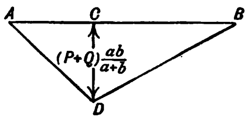


Fig. 197.

S. F.

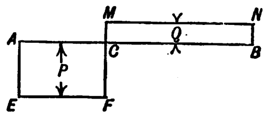


Fig. 198.

Case III—Beam fixed at one End and loaded at the Other.—This is the case which we first used to explain B. M. and S. F.

Taking K any point whatever,

$F_K = W$, and $M_K = W . BK$.

This holds everywhere, hence

$F_A = W$, $M_A = W . BA = Wl$.

Now, does this contradict what we have said in Case I as to there being no possible action on the free end of a beam? The answer is No! Because A is not a free end, nor is it really the end at all, for some part of the beam would be fastened into the wall. AB is, however,

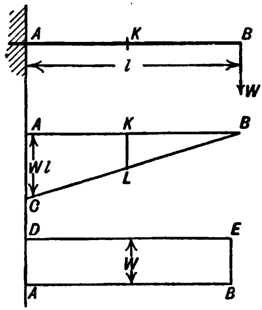


Fig. 199.

called the length of the beam, and A is spoken of as the end; but we say it is not free but is acted on by the wall.

In this case we *must* take the forces on the right of K; because we have no knowledge of those on the left, *i.e.* the actions of the wall, except that they are such as to balance W, and so we cannot find *all* the forces in this case. The graphic construction is simpler than Case I.

For at all points, as K, M varies directly as BK.

So the curve of B. M. is a straight line BC, drawn by making AC represent on some selected scale Wl lbs.-feet of moment, *i.e.* to represent M_A.

Then we can prove, as in Case I, that the ordinate KL represents M_K.

The S. F. being everywhere equal to W, we take AD to represent W and draw DE parallel to AB. Then DE is the curve of S. F.

Notice that the bending is now minus, so AC is below AB. The shearing is, however, plus everywhere.

Case IV—Beam supported at the Ends and loaded at two Intermediate Points.—The loads are W_1, W_2 at C and D (Fig. 200).

First to find P and Q.

Take moments about B, then we have

$$P \cdot AB = W_1 \cdot BC + W_2 \cdot BD,$$

$$\therefore P = \frac{W_1 \cdot BC + W_2 \cdot BD}{AB}.$$

Similarly

$$Q = \frac{W_1 \cdot AC + W_2 \cdot AD}{AB}.$$

Then for K_1 in AC (not marked in the figure)

$$M_{K_1} = P \cdot AK_1, \quad \therefore M_C = P \cdot AC,$$

and the curve is the straight line AE, where

$$CE = P \cdot AC.$$

Similarly for BD the curve is BF, where

$$DF = M_D = Q \cdot BD.$$

Now for a point K between C and D. There is no choice of sides, suppose we take the left, then

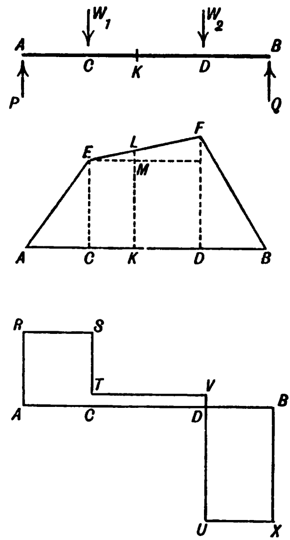


Fig. 200.

$$M_K = P \cdot AK - W_1 \cdot CK,$$
$$= P \cdot AC + (P - W_1)CK \ (1).$$

If then we draw $KL = M_K$ (Fig. 200), and draw EM parallel to AB; we have

$$KL = KM + LM = CE + LM.$$

But $CE = P \cdot AC$,

$\therefore LM = (P - W_1)CK$ (from 1).

Therefore LM/CK or LM/EM is constant for all positions of K, so that L lies on a straight line through E.

Evidently exactly similar reasoning would prove that L must lie on a straight line through F. Hence we conclude that L lies on EF, and therefore AEFB is the diagram of B. M.

Next for the S. F.

Commencing at A, between A and C, F is $+P$, and the curve is RS parallel to AC.

Between C and D we have to the left of any point $P - W_1$ upward; now we notice that in Fig. 200 we have drawn L above E, which assumes that $P - W_1$ is positive; therefore F is $+(P - W_1)$ between C and D, and the curve is TV where $CT = P - W_1$ or $ST = W_1$.

In BD we have $P - (W_1 + W_2)$, which is negative, on the left, or $+Q$ on the right; F then is $-Q$, and we set off Q downwards, the curve then being UX. Thus RSTVUX is the full curve of shear.

The student may for practice commence at B and work to the left; he should thus obtain the same curve.

We have worked this question by the preceding method simply to show how, by keeping strictly to the definitions, we are able to find the value, and hence draw the curves, of B. M. and S. F. at every point. Generally, however, the simpler way will be to use a method which will be explained in the next chapter.

EXAMPLES.

1. A plank is laid across an opening 12 feet wide. A man weighing 156 lbs. walks across it. Draw diagrams showing the S. F. and B. M. when he is 2, 4, and 6 ft. respectively from one end. Give the maximum values in each case.

Ans. 130, 104, 78 lbs.; 260, 416, 468 lbs.-feet.

2. A similar plank to that of (1) is broken by the bending effect of 3 cwt. placed 3 ft. from the end. What is the greatest load the man could safely carry across? *Ans.* 96 lbs.

3. The air pump of an engine is driven by a lever or rocking arm, total length 8 feet. The air pump stroke is half the piston stroke. Find the greatest bending moment on the lever, when lifting the pump bucket 18 ins. diameter against an effective pressure of 16 lbs. to the square inch. *Ans.* 10,862 lbs.-feet.

4. The speed of periphery of a spur wheel is 20 f.s. The teeth are $1\frac{1}{2}$ in. long. Find the bending moment at the root of a tooth, for each H. P. transmitted, assuming only one tooth in gear at a time, and the whole pressure to come on the point of the tooth. *Ans.* $41\frac{1}{4}$ lbs.-inches.

5. A loaded truck weighing 10 tons rests on two axles. The axles are supported by the wheels 5 ft. apart, and the centres of the axle boxes are 4 ft. 4 ins. apart. Draw curves of B. M. and S. F., and give numerical values.

Ans. B. M.—0 at ends to 10 tons-inches at and between axle boxes; S.F.—$\frac{5}{2}$ ton from ends to boxes, 0 between boxes.

6. The horse-power of an engine is 100. Stroke, 4 feet. Revolutions, 50 per minute. Find the bending moment on the crank arm at its junction with the shaft, the diameter of the latter being 6 ins. *Ans.* 14,437.5 lbs.-feet.

7. A beam, l feet long between supports, overhangs a ft. and b ft. at the two ends respectively. At the first end a load W_1 hangs, and at the second a load W_2. Show from the definition that the bending moment midway between the supports is $\frac{W_1a + W_2b}{2}$, and is thus independent of l. Draw the curves of B. M. and S. F.

8. The top of a combustion chamber 2 ft. 8 ins. deep is supported by rows of three stays, each stay carrying a load of 6000 lbs., spaced at equal distances apart and from the front and back plates. Each row is supported by a "girder" or "dog," the ends of which rest on the front and back plates. Find the bending moment on a girder at each of the three points where it carries a stay. *Ans.* 6000, 8000, 6000 lbs.-feet.

CHAPTER XV

B. M. AND S. F. UNDER DISTRIBUTED LOADS—PRINCIPLE OF SUPERPOSITION

In the last chapter we considered the effects of loads acting at points, qualified by explaining that "at a point" really means over a small length. The work then applies to cases in which the load carried is concentrated on one or more small lengths of the beam, *e.g.* a locomotive on a bridge.

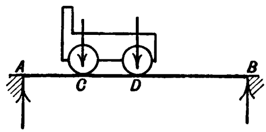


Fig. 201.

Here the total weight is concentrated at the two points C and D, in proportions depending on the distribution of weights in the locomotive. In Fig. 201 the engine is of appreciable length compared to the bridge; but if the engine were, say, on one of the spans of the Forth Bridge, then, for all practical purposes, it would be quite sufficient to consider the engine as one weight only, concentrated at its C. G.

But now take the case in which a bridge is covered by a densely packed crowd; then the points of application of the loads are so numerous that practically there is a continuous load at every point of the bridge. One kind of load we can see is perfectly continuous, viz. the weight of the beam itself.

Such loads as we have just mentioned are called Distributed Loads; and if they be so distributed that the

load on every unit length is the same, they are then said to be uniformly distributed. In the present work this latter is the only kind we shall consider.

Intensity of Load.—The intensity of the loading is generally estimated by the load on each foot length, or running foot, of the beam; so that it will be stated as so many lbs., cwts., or tons *per foot run*.

For example, a bridge 20 ft. span, loaded with a uniformly distributed load of 10 tons, would be said to be under a loading of $\frac{1}{2}$ ton per foot run. The word *span* here used signifies the length between the supports.

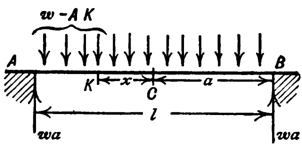


Fig. 202.

We will now investigate some cases of such loading.

Case V — Beam supported at the Ends, and loaded with w lbs. Per Foot Run.—Let the span AB be denoted by l or $2a$—the latter being often useful to avoid fractions. These must be in feet. Then

$$\text{The total load on the beam} = wl, \quad \text{or } 2wa.$$

This we generally denote by W,

$$\therefore \; W = wl, \quad \text{or } 2wa.$$

First we must find the supporting forces. These are evidently equal, and each is, therefore, $W/2$ or wa.

The uniform loading is represented by the small arrows, but its amount is not indicated in the figure.

Take C the centre, and take any point K, K being defined by its distance CK from the centre. This distance we call x. Then

$$\begin{aligned} M_K &= \text{moment of forces to the left of K about K,} \\ &= wa \,.\, AK - \text{moment of load on AK.} \end{aligned}$$

This latter consists of an infinite number of small forces, but we can find its moment, because, by a principle of Statics,

$$\begin{matrix}\text{The moment about any point} \\ \text{of a set of forces}\end{matrix} = \begin{matrix}\text{The moment about the point} \\ \text{of their resultant.}\end{matrix}$$

The resultant of the load on AK is a force w . AK, acting through the centre of AK.

$$\therefore \text{ moment of resultant} = w \,.\, \text{AK} \times \frac{\text{AK}}{2} \text{ about K.}$$

$$\therefore \ M_K = wa \,.\, \text{AK} - \frac{w \,.\, \text{AK}^2}{2}.$$

But $\text{AK} = a - x$,

$$\therefore \ M_K = wa(a-x) - \frac{w(a-x)^2}{2},$$

$$= \frac{w}{2}(a^2 - x^2).$$

We must here call particular attention to a constantly occurring source of error in these cases.

In finding M we have to estimate the moment of the loads on AK by taking the moment of their resultant. Then the mistake is constantly made of putting the resultant in the figure, and treating it as if it actually could replace the loads.

For example, we wish say to find M_C. Then

$$M_C = wa \,.\, a - wa \,.\, \frac{a}{2},$$

and then the resultant force wa is drawn in, as in Fig. 203, and the problem is treated as if this force actually acted at E. Let us see now what result this would give us at K. We should have

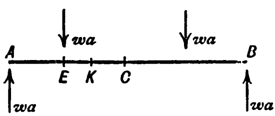


Fig. 203.

$$M_K = wa \,.\, \text{AK} - wa \,.\, \text{EK},$$

$$= wa \,.\, \frac{a}{2},$$

which is of course wrong.

The whole source of the error lies in supposing that the resultant of a set of forces can replace them for

all purposes, whereas it can only replace them for one, viz. the movement, as a whole, of the body acted on. Thus for M_C, the body whose movement is prevented is AC, and so far as M_C is concerned, wa may be spread over AC or concentrated at E. But for M_K the body is AK; and the part of wa which lies along CK does not act on this at all, so that it is plainly wrong to take a force wa acting at E.

The student must remember that in abstract mechanics the motion of a body, as a whole, is generally the thing considered; while here we are seeking to find the relative movements of the parts of the body.

In all cases then in which we require to deal with a set of forces such as those on AK (Fig. 202), it is conducive to clearness not to draw in their resultant, but if it be desired to indicate in some way the set with which we are for the moment dealing, it is best to do it by drawing a bracket over them, and marking their total amount on it as is done in Fig. 202. Resuming now our calculation; we had

$$M_K = \frac{w}{2}(a^2 - x^2).$$

For the point C we put $x=0$, whence

$$M_C = \frac{wa^2}{2},$$

$$\therefore\ M_K = M_C - \frac{wx^2}{2}.$$

We put it this way to facilitate the consideration of the shape of the B. M. curve (Fig. 204).

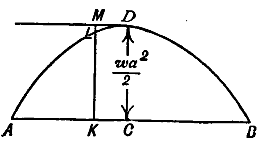


Fig. 204.

For set up $CD = M_C = \frac{wa^2}{2}$, and KL to represent M_K. Draw DM parallel to AB, and produce KL to M. Then

$$ML = M_C - M_K = \frac{wx^2}{2}.$$

Therefore L lies on a curve such that the distance of

any point in it from DM varies as the square of its distance from CD. This curve is a parabola (page 6). And since

$$M_A = M_B = 0,$$

the curve of B. M. is a parabola, on AB as a base, D being its apex. The height CD may be variously expressed as

$$\frac{wa^2}{2}, \quad \text{or } \frac{wl^2}{8}, \quad \text{or } \frac{Wl}{8}.$$

Looking at the latter value we see that the maximum bending moment produced by a distributed load is only one half of that which would be produced if the load were concentrated at the centre of the beam.

To actually draw the curve ADB, we first set up CD $= \frac{1}{8}Wl$, and then use the method of page 6.

Next for the S. F.,

Looking to the left (Fig. 205),

$$F_K = w \,.\, a - w \,.\, AK,$$
$$= w \,.\, x.$$

For C then the curve is a straight line passing through C and D, where D is found by making AD $= wa$. The shear at K being plus, we draw CD above AC.

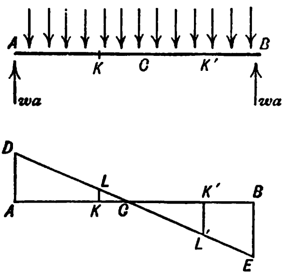


Fig. 205.

At a point K′ in CB, at a distance x from C, we shall have

$$F_{K'} = wa - w \,.\, BK' \quad \text{(looking to the right)},$$
$$= wx.$$

So it is the same numerically as F_K, but being a case of right hand rising, its sign is minus.

If then we produce DC to E, DCE will be the curve of S. F. for the whole beam. For

$$BE = -AD = -wa = F_B,$$

and

$$K'L' = -wx = F_{K'}.$$

[If we compare this curve with those in the preceding chapter, we see that we have replaced steps by a continuous incline, or succession of indefinitely small steps, this being the effect of the distribution of the load.]

Case VI—Beam loaded uniformly, fixed at One End.—AB is fixed at A, length l ft., loaded with w lbs. per foot run (Fig. 206).

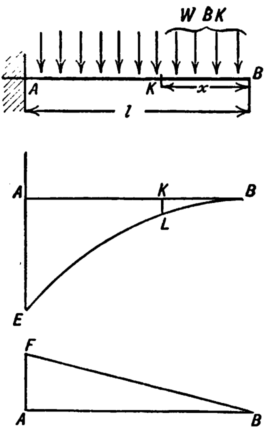


Fig. 206.

We have now no supporting force to find.

Take then any point K. Let $BK = x$. Then

$$M_K = w \,.\, BK \,.\, \frac{BK}{2} \text{ (compare page 295)},$$

$$= \frac{wx^2}{2},$$

and the bending is negative, the convex side being upward (page 284). Set off

$$KL = M_K = \frac{wx^2}{2}.$$

But we see now that KL has the identical value which ML in Fig. 204 had.

Hence then we say at once that L lies on a parabola, whose apex is B and base parallel to AB. To find another point we find the value of M_A. Putting $x = l$,

$$M_A = \frac{wl^2}{2},$$

and is minus as M_K is.

So then set off AE downwards, then E is a point in the parabola, and it is then drawn by the rule given in the Preliminary Chapter.

[Notice carefully the difference between this and the preceding curves. In the present case the base from which the moments are measured is not what in the Preliminary Chapter we called the base, nor is it the base as in the last case. In the present case the apex is on the base for measurements, and the convexity is towards that base; while in the preceding the apex was above the measurement base, and the concavity was towards it.]

The calculation of S. F. is very simple. For

$$F_K = w \cdot BK = wx.$$

Thus the shear at A is wl, and the curve is the straight line BF, drawn by setting up $AF = wl$, since the shear everywhere is plus, and joining BF. Then

$$KL = AB \cdot \frac{x}{l} = wl \cdot \frac{x}{l} = wx = F_K,$$

which verifies the curve.

Platform Loads—Weight of Beam.—In most, perhaps all, cases the load is distributed not only along the length of the beam, but also in a direction perpendicular to the beam—one or more beams being used to support a loaded platform. We should be given the load per sq. ft. of this platform, and we wish to determine from this the load per foot run on the beam.

Take now the case of a bridge platform, span l ft., width b feet, loaded with n lbs. per sq. ft. Generally such a platform would be supported by two beams of span l, one under each side.

Taking then one foot length of the bridge, its load will be divided equally between the two foot-lengths of the beams. Therefore, if w be the load per foot run on each beam,

$$2w = n \times 1 \times b,$$

$$\therefore\ w=\frac{nb}{2} \text{ lbs. per foot run.}$$

If, in addition, the beam itself weighed w' lbs. per foot run, we should have finally,

$$w=\tfrac{1}{2}nb+w' \text{ lbs. per foot run,}$$

and we must use this value of w to find the B. M. and S. F.

In the general case, where a number of parallel beams support a loaded platform, the load on each beam would be found by supposing it to support the piece of platform which extends on each side of it, half way towards the next beam. Thus in Fig. 207 AB, CD, etc., are the beams shown in plan, and CD would support the piece of platform dotted; and the total load on this piece, divided by the length of CD, would give the load per foot run.

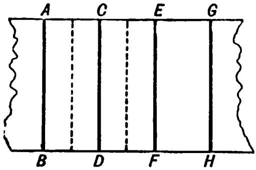


Fig. 207.

Floating Beams—Case VII.—In some cases the distributed loads may act upwards, or, in other words, the supporting forces may be distributed. Such a case is that of a wooden beam floating in water.

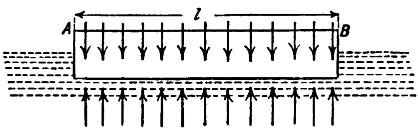


Fig. 208.

Taking the beam alone, there is on every foot-length a distributed load w' lbs., where w' is the weight of 1 foot. But the upward pressure of the water will also be uniform, and must therefore be also w' per foot run. At

every point, then, there are equal and opposite forces acting, so there is no tendency to bend or shear.

But now let us place a load $W = wl$ pounds on the centre of the beam, then the beam sinks until the increase in the upward pressure is sufficient to balance W or wl. Since W is put on the middle the beam sinks evenly, and there is therefore a uniform increase of pressure over the whole length.

[The case in which the loading is not symmetrical is outside our limits.]

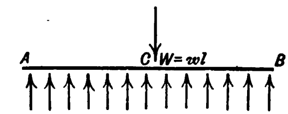


Fig. 209.

This increase of pressure, then, is w lbs. per foot run.

The beam is then in equilibrium under a downward load W lbs., an upward distributed load w lbs. per foot run, and equal and opposite distributed loads w' lbs. per foot run. Since these latter, however, produce no B. M. or S. F., we can omit them, and we are left with the beam loaded, as in Fig. 209.

To make it more easily recognisable we turn it upside down (Fig. 210); this cannot affect the values of M or F, and for their signs we can look to the original figure. We appear now to have a case quite different from any of the preceding, but by a little consideration we shall be able to make use of previous results.

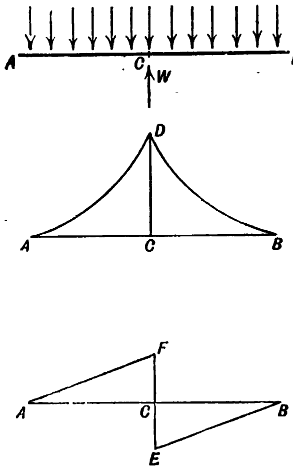


Fig. 210.

For consider the part CB, then CB is a beam *fixed at* C, and loaded uniformly. Now this may at first sight appear a somewhat strange statement, but it is nevertheless perfectly correct. In Fig. 206 AB is fixed to a wall, but then the question as to what sort of wall it was did not enter into our work; what we require in order to be able to fix the end A is something which is capable of standing M_A and F_A without giving way, and so long as it can do that we care nothing else about it. Now CB is fixed at C to the other half of the beam, and the fixing is strong enough to withstand M_C and F_C, because if it were not the beam would break, which we suppose it not to do. Hence then we treat CB as a beam fixed at C, and we can at once draw the curve for CB from Case V. We set up then CD (the bending being plus, see Fig. 191) equal to $w.CB^2/2$, and draw the parabola DB with apex B.

Then from symmetry a parabola AD will be the curve for AC, that being a beam also fixed at C.

Hence ADB is the curve of B. M., CD being $wl^2/8$ or $Wl/8$.

In BC the shearing force is negative (Fig. 192), therefore from Case V set down

$$CE = w.CB = \frac{W}{2},$$

and join BE.

In AC the shearing is positive (Fig. 192), therefore from Case V set up

$$CF = w.CA = \frac{W}{2},$$

and join AF.

Then AFEB is the full curve.

[For the explanation of the drop at C see page 285.]

The bending and shearing of a ship is a similar case to the preceding; but there is an uneven distribution both of load and of buoyancy or upward pressure,

for the effect of which we must refer to the larger treatise.

Principle of Superposition.—When a beam is loaded with more than one load, we can, as in Case IV of the last chapter, obtain the curves of B. M. and S. F. by strict adherence to the definitions; this is also true if one or more concentrated loads be combined with a distributed load. But the process can generally be considerably adopted by the use of what is known as the Principle of Superposition.

This principle may be generally stated thus: *The effect due to a combination of causes is the sum of the effects which would be produced by each cause acting separately.*

This may, at first sight, appear to be a truism, but it is not so, for it is only true under certain conditions. To show this, and also what are the conditions, we will consider a case.

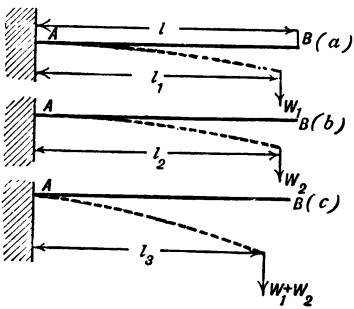


Fig. 211.

Suppose AB a beam fixed at A, and so thin that the hanging to B of a weight W_1 bends it as Fig. 211 (*a*), so that the distance l_1 of B from the wall is perceptibly less than l, the original length. Similarly in (*b*) a weight W_2 bends it, so that B is distant l_2 from the wall.

Plainly then if we put on both W_1 and W_2 the distance l_3 in (*c*) will be less that either l_1 or l_2. Now

$$M_A \text{ due to } W_1 \text{ and } W_2 = (W_1 + W_2) l_3,$$

which is not the sum of $W_1 l_1$ and $W_2 l_2$, the moments at

A produced by W_1 and W_2 acting separately, but is plainly less.

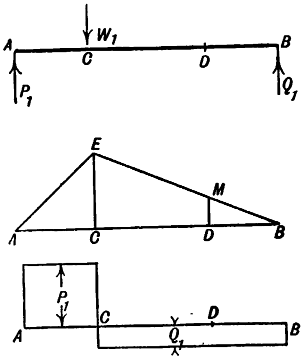


Fig. 212.

We see, then, that what is necessary for the principle to apply is that l_1, l_2, l_3 should be practically identical, and then they will also each be identical with l.

Generally we may say that all the separate effects should be very small.

Now it is proved both theoretically and practically that in all ordinary cases of bending, the bent lengths l_1, l_2, and l_3 (Fig. 211) are indistinguishable from the original length l, and so the principle holds. For

$$M_A = (W_1 + W_2)l = W_1 l + W l,$$
$$= \text{M due to } W_1 \text{ alone} + \text{M due to } W_2 \text{ alone.}$$

We will now illustrate the principle by using it for Case IV of the last chapter.

Case IV Repeated.—We take first a beam AB supported at A and B, and loaded at C with the weight W_1.

The supporting forces are P_1 and Q_1, and Fig. 212 shows the diagrams of B. M. and S. F. from Case I. The values are

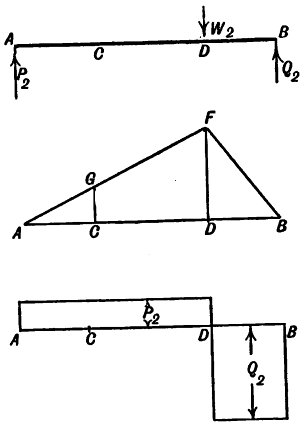


Fig. 213.

$$P_1 = W_1 . \frac{BC}{AB}, \quad Q_1 = W_1 \frac{AC}{AB},$$

and the height $CE = \frac{W_1 . AC . CB}{AB}$.

Next take W_2 alone (Fig. 213).

Then by again applying Case I we get the diagrams of Fig. 213. And

$$P_2 = W_2 . \frac{BD}{AB}, \quad Q_2 = W_2 . \frac{AD}{AB},$$

and

$$DF = W_2 . \frac{AD . DB}{AB}.$$

Now combine the two sets of results. Then the total supporting forces are given by

$$P = P_1 + P_2 = W_1 . \frac{BC}{AB} + W_2 . \frac{BD}{AB},$$

$$Q = Q_1 + Q_2 = W_1 . \frac{AC}{AB} + W_2 . \frac{AD}{AB}.$$

Compare these and also all the succeeding results with those obtained in the last chapter.

Next, for the combined curve of BM, add the two curves together thus :—

Draw AEB from Fig. 212; then at E set up EN = CG (Fig. 213); and at M set up MS = DF (Fig. 213); and then joining ANSB, the curve ANSB represents the sum of the curves AEB and AFB (Fig. 213). We add them together, because they are both positive, and we obtain a positive result.

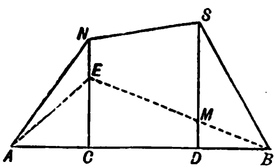


Fig. 214.

A quicker way, however, is to set off AFB on the opposite side of AB to AEB; and then obtain the moment at any point by measuring right across. Thus at K (Fig. 215)

$$M_K = VU.$$

We have now no base line, and no guide as to the sign of the bending, but this latter is generally quite plain from the original figure.

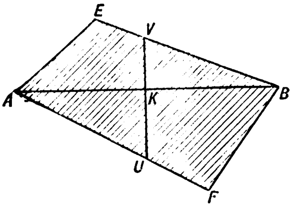


Fig. 215.

Now for the S. F.

We have to add Figs. 212 and 213. We draw then in Fig. 216, Fig. 212, dotted, and proceed to add to it Fig. 213.

Between A and C we have in each case + shear, so we set off P_2 above P_1; also between B and D both are −, and so we set off Q_2 below Q_1. Between C and D we have a + and − to add. So Q_1 being −, we add to it P_2+, by setting P_2 upward, and P_2 being greater than Q_1, the sum is +; this is shown by the full curve coming above AB. This curve is exactly that on page 289, for the height along CD is now $P_2 - Q_1$. And

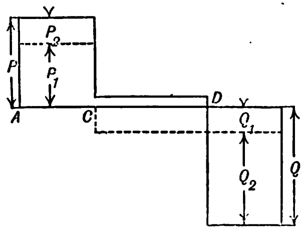


Fig. 216.

$$P_2 - Q_1 = (P - P_1) - Q_1 = P - (P_1 + Q_1),$$
$$= P - W_1,$$

which is the height given in Fig. 200.

In the curve just drawn all ordinates are measured from the base line AB. But we can add the curves of S. F. by drawing them on opposite sides of AB, and measuring across as we did for B. M.

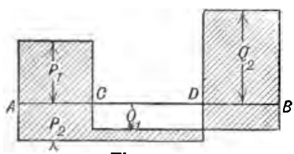


Fig. 217.

We thus get Fig. 217, and the S. F. at any point is given by the breadth of the shaded figure.

The second method of adding has the advantage of being automatic, but we lose our base line, so the curve does not indicate the sign of the shear; also we can in this way add only two results together, while the first method is applicable to any number.

The common form of error in the application of superposition is to first find P and Q, the total supporting forces; and then to use P, W_1, and Q as one set of forces, and P, W_2, and Q as another set. This is of course erroneous; because superposition requires us to examine the separate effect of each load, and the effect of W_1 is not to produce P and Q but only P_1 and Q_1. The production of P and Q is a joint effect.

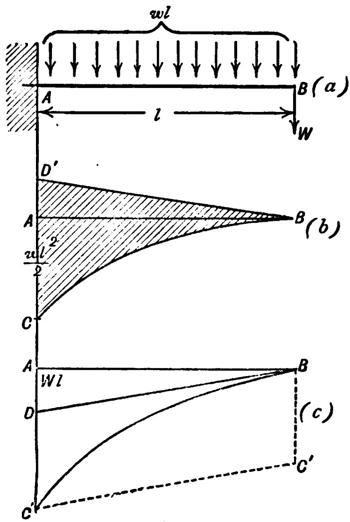


Fig. 218.

Case VIII.—We will next consider a combination of distributed load with concentrated load. AB is fixed at A, loaded uniformly, and also with a single load at B. Fig. 218 (*a*) shows the loading. Then in (*b*), taking $AC = wl^2/2$, the parabola BC is the curve of B. M. for the distributed load only; and in (*c*), taking $AD = Wl$, the line BD is the curve of B. M. for W only.

They can then be added, either as in (*b*) by drawing AD′B representing ADB inverted, and measuring across the shaded area, or as in (*c*), by keeping AB as a

base and adding the ordinates of CB to those of BD, thus obtaining the curve C′B. It is easy to show by actual construction that C′B is also a parabola, constructed on C′C′ as half base, and its greatest height from that base being at B (page 7).

For the S. F.

Fig. 219 (*a*) shows CB, the curve for the distributed load; and (*b*) shows DE, the curve for W. Then in (*a*) the two curves are added by inverting DE as D′E′; while in (*b*) the ordinates of CB are added to those of DE, giving the total curve C′E. The results are identical.

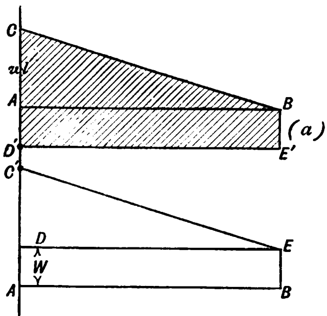


Fig. 219.

Innumerable other combinations can be made, but enough has been done to show the application of the methods; and other cases will be left to the student to solve for himself, by one of the modes of the present chapter.

Examples.

1. A timber beam is 18 feet between supports, and is 12 ins. deep by 4 ins. broad. Draw curves of B. M. and S. F. produced by its own weight, giving numerical values at each quarter span. Weight of the timber, 48 lbs. per c. ft.
   Ans. F = 144, 72, 0, 72, 144 lbs.; M = 0, 483, 648, 483, 0 lbs.-ft.

2. Allowing a bending moment of 6800 lbs.-ft., how far apart should such beams be spaced when supporting a floor loaded with 40 lbs. per sq. ft.? *Ans.* 3 ft. 9 ins.

3. A balcony projects 6 ft., and is supported by beams spaced 6 feet apart. Find the bending moment at each foot length when the balcony is loaded with 120 lbs. per sq. ft.
   Ans. 360, 1440, 3240, 5760, 9000, 12,960 lbs.-ft.

4. Obtain the results of questions 5 and 7, pages 291 and 292, by the principle of superposition.

5. An oak beam 15 ft. long, 1 ft. square, floats in sea-water and is loaded at the centre with a weight just sufficient to immerse it wholly. Draw the curves of B. M. and S. F., giving their maximum values. 35 c. ft. of sea-water weighs 1 ton. 1 c. ft. of oak weighs 48 lbs. *Ans.* 120 lbs.; 450 lbs.-ft.

6. If, in question 3, the end of each beam were supported by a pillar which carried one-third of the whole load on the beam; find the results.

Ans. – 1080, – 1440, – 1080, 0, 1800, 4320 lbs.-ft.

7. The steel crank-shaft and pin of a vertical engine is 12 ins. diameter, each crank arm is 14 ins. by 8 ins., the distance from centre to centre of bearings is 50 ins., the crank pin length being 15 ins., stroke 4 ft. The thrust of the piston rod is 45 tons, and may be taken as applied to the centre of the pin. Draw the curves of B. M. and S. F., and give values at the centre.

Ans. There is a continuous load due to the shaft and pin, plus the extra load of the arms. Take the latter as acting in the centre line of each arm. Then distributed load is 380 lbs. per ft. run, load at centre 100,800 lbs., extra load due to each arm 870 lbs., therefore maximum B. M. = 106,800 lbs.-ft.

8. A bar of iron 1 inch diameter can only withstand a moment of .4 tons-inches. Determine the greatest length of bar which can just carry its own weight when supported at the ends.

Ans. 141 ft.

9. A piece of plate 8 ins. broad, 2 ft. long, is supported by 3 stays, one at the centre, and one 8 ins. from the centre on each side. The plate is subject to a pressure of 130 lbs. per sq. inch. Assuming that each stay carries the same load; find the bending moment at each stay, and draw the curve of S. F.

Ans. $693\frac{1}{3}$ lbs.-ft.

CHAPTER XVI

OPENWORK BEAMS—WARREN AND N GIRDERS

THE beams which carry the platforms of bridges, etc., may be solid, or, more generally, built up of bars and plates connected together by pins or rivets. They are then usually called **Girders**, or, when consisting of a network of bars, **Trusses**; and since the manner in which the B. M. and S. F. are resisted, and the nature of these actions, is shown more clearly in open work than in solid beams, we will take them first in order.

Fig. 220 shows such a girder in skeleton. There

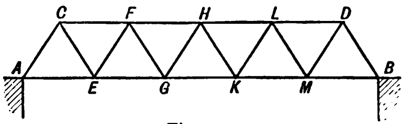


Fig. 220.

are two horizontal members, AB and CD, called the **Booms** or **Flanges**. These booms are connected by cross bars AC, CE, etc., called the **Diagonals**, or collectively constituting the **Web**.

In the figure the top and bottom booms are parallel, and hence the girder is called a **Parallel Girder**; in many cases either CD or AB may not be straight and parallel, but these cases we shall not discuss at present.

In the figure also the diagonals do not cross each other, or we say the web is a simple triangulation; in

many cases this is not so (see next chap.), but in the present chapter we shall deal only with such a simple triangulation. There are two main forms which the simple parallel girder takes, viz. the form of Fig. 220, the diagonals being inclined at 60° to the horizontal, known as the **Warren Girder**; or the form shown in Fig. 221, with diagonals alternately upright and sloping at 45°, known as the **Linville** or **N Girder**. In this case the uprights are not usually called diagonals but verticals; "diagonals," however, may be used when we wish to denote generally any bar of the web.

Pin Joints and Riveted Joints.—There are two distinct ways in which a girder may be constructed:

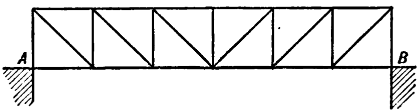


Fig. 221.

First, the booms may consist (Fig. 220) of separate bars—CF, FH, etc., and AE, EG, etc.—jointed by pin joints to each other and to the diagonals, no bar being continuous through a joint. Or—

Second, the booms may be continuous bars, or the joints all riveted so as to make the structure practically one continuous whole.

For reasons explained fully in chap. xxii. (page 437) we shall confine our attention to the first case. Our girder then is supposed made up of separate bars united by pin joints; the joints being supposed frictionless, so that they can offer no resistance to turning.

Loads at the Joints.—In practically all cases the loaded platform carried by the girder or girders rests not on the bars of the girder but on cross beams, which cross beams rest on the boom or booms at the joints.

Thus Fig. 222 (a) represents the plan of a bridge,

this is carried by a pair of girders shown in plan by AB, A′B′, one under each side of the bridge. Then 11′, 22,′ 33′ represent in plan the cross beams which would carry the platform; these in turn resting on the joints 1, 2, 3 of AB, shown in elevation in (*b*), and the corresponding joints of A′B′.

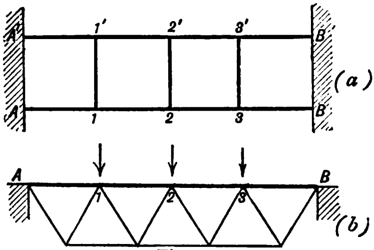


Fig. 222.

The girder then, as distinguished from the platform or the bridge as a whole, is loaded, not with a distributed load, but with loads concentrated at the joints. The loaded joints may be those of the top boom, as Fig. 222, or of the bottom boom, or of both; also the girder may be supported at the two ends, or as in the case of a balcony, fixed at one end only. We will proceed to examine some of these cases.

Warren Girder fixed to a Wall.—Taking a general case, let it be loaded as shown at the joints 1, 2, and 7, with the loads W_1, W_2, and W_3. The joints are denoted by numbers, which is found more convenient than using letters.

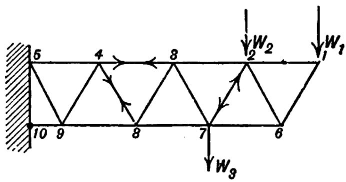


Fig. 223.

We want now to find the stresses in the bars, and we must begin by finding for any given bar what duty it performs. The simplest way of doing this is to imagine the bar removed, and see what effect would be produced.

Action of Booms.—Consider then the bar 34.

Imagine it removed, then evidently (Fig. 224) the part to the right swings downward round the joint 8, *i.e. the joint opposite to* 34.

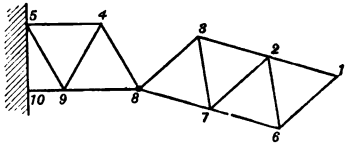


Fig. 224.

It appears then that the duty of 34 is to prevent rotation round the joint 8. This it effects by preventing the separation of 3 and 4, which is necessary if the rotation take place. Since, when 34 is removed, 3 and 4 would separate, or the distance 34 would increase, it follows that when 34 is in place there is a *tendency* to elongate 34. Now we know from chap. xiii. that such a tendency will be resisted by the internal stresses on the cross-sections of the bar, or that 34 tries to regain its original length; and in so doing it exerts pulls on the pins of the joints at 3 and 4, these pulls being equal and opposite, and each representing the total stress on any section (page 262). Let us denote this total stress in 34 by H_{34}, then 34 pulls at each of the points 3 and 4 with a force H_{34}, and this is represented in Fig. 223 by placing the arrows on the bar, these arrows pointing inward, and thus representing the force exerted *by the bar on the joints.* This is a point of the first importance, and the arrows should always be put in this manner; if we wished for any reason to denote the forces acting on the bar, they would be put thus

$\leftarrow\overset{4}{\text{———}}\overset{3}{}\rightarrow$. It is advisable to put the arrow near the joint on which the force it represents acts; and always to put in the pair of arrows, not one only; then the action of each bar is plain, *e.g.* 34 in Fig. 223 evidently pulls 3 to the left and 4 to the right.

Next, consider the effect of removing a bar of the bottom boom, say 89. Then the right-hand part swings down round 4, *the joint opposite to* 89.

It appears then that the function of a bar in either boom is to resist rotation *round the opposite joint.*

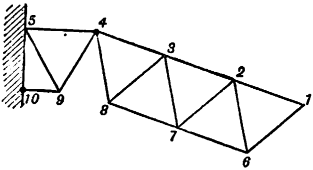


Fig. 225.

There is, however, a difference in the nature of the stresses in the two cases, for whereas 34 elongated, we now find that 8 and 9 tend to approach and are thrust apart by the forces in 89; hence the arrows point outward, and H_{89} is the resistance of 89 to compression. We see then that in this instance the bars of the top boom are extended, and of the bottom boom compressed.

Next, we want to find the magnitudes of H_{34}, H_{89}, etc.

Take now H_{34}, then to find a connection between it and the loading, we must consider the equilibrium of some body on which H_{34} acts. We have (Fig. 223) a choice of two bodies, one being the piece 8, 3, 2, 1, 6, 7, which H_{34} pulls to the left, and the other the body 5, 4, 8, 9, 10, which it pulls to the right. We select the first because we know more about the forces acting on it. Fig. 226 shows the piece 8316 taken out separately. This is now a body in equilibrium under W_1, W_2, W_3, H_{34}, and the forces which act at the joint 8. These latter we do not know, but we can avoid considering them by taking moments about 8. This gives us

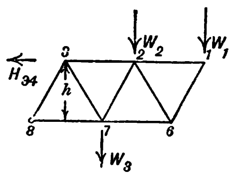


Fig. 226.

Moment of H_{34} about $8 =$ moments of W_1, W_2, and W_3 about 8.

But the quantity on the right-hand side is what we have defined as the B. M. at 8, and, moreover, this

holds not only for our present loading, but for any number of W's. If, then, we call the depth of the girder h (Fig. 226), we have

$$H_{34} \cdot h = M_8.$$

The work above applies exactly to any other bar and the joint opposite, *e.g.* take 23.

Then H_{23} keeps 7261 in equilibrium, and taking moments about 7,

$$H_{23} \cdot h = \text{moments of } W_1 \text{ and } W_2 \text{ about } 7, \\ = M_7.$$

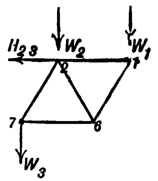


Fig. 227.

Or, taking the bottom boom, H_{89} keeps 4168 in equilibrium, and taking moments about 4,

$$H_{89} \cdot h = M_4.$$

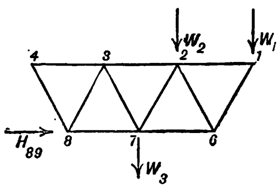


Fig. 228.

Hence, then, we say that in all cases

$$Hh = M,$$

where H is the stress in any bar, and M is the B. M. at the opposite joint. By the opposite joint we mean this one about which rotation takes place if the bar is removed, *i.e.* the apex of the triangle of which the bar is the base. In order to determine whether H be tension or compression, we must consider whether the bar keeps the joints at its ends together, or whether it thrusts them apart; in the first case we have tension, in the second compression.

Since we have not in the work used any distinctive property of the Warren girder, the result will also apply to any system of simple triangulations whatever.

Action of Diagonals.—Referring to Fig. 223, let us now suppose one of the diagonals, say 48, removed. Then the girder takes the form of Fig. 229. Hence the office of 48 is to prevent the part 8316 from falling bodily, keeping parallel to itself. The bar effects this by

means of its resistance to extension, so that 48 exerts on the joints 4, 8 equal forces, say S_{48}, pulling the joints

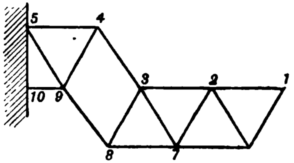


Fig. 229.

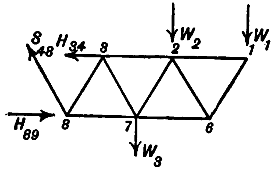


Fig. 230.

together. This we show in Fig. 223 by the arrows (compare page 313), and S_{48} is the tension in 48.

For the magnitude of S_{48}, consider the equilibrium of 8316.

[We may here notice that, although we suppose the girder to distort in order to see what sort of action there is, yet we must not take the distorted position to consider the equilibrium, because that is not the position in which the forces hold the piece, but they hold it in its original position.]

8316 is now in equilibrium under W_1, W_2, W_3, H_{34}, H_{89}, and S_{48}.

We have here six forces, but by resolving vertically we can get rid of the H's; representing the angle the diagonal makes with the vertical by θ, this gives

$$S_{48} \cos \theta = W_1 + W_2 + W_3.$$

But $W_1 + W_2 + W_3$ is the S. F. for any cross-section between 4 and 8, *i.e.* for any cross-section cutting the bar 48. We have then

$$S_{48} \cos \theta = F_{48},$$

F_{48} meaning as just stated.

We will now take a bar sloping in the other direction, say 27.

Then on removal, Fig. 231 shows the form assumed, hence we see 27 is in compression, and by its resistance thrusts 2 and 7 apart. This is shown by the arrows.

For the magnitude of S_{27} we have, by the same method as before,

$$S_{27} \cos \theta = W_1 + W_2 = F_{27}.$$

θ is the same as before for the Warren girder, but

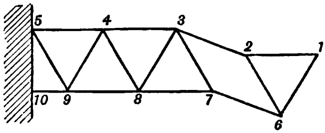


Fig. 231.

it may in some cases be different for different bars; this will not affect the equation.

We see then that just as

$$Hh = M$$

gives the stress in any bar of the booms, so

$$S \cos \theta = F, \quad \text{or } S = F \sec \theta$$

gives the stress in any diagonal, noting, however, that M is at a point, while F is the value of the S. F. at any point of the length covered by the particular diagonal considered.

[It is important to note this difference, because if we give F at a point where a weight acts, there comes in a difficulty as to the weight at the point. For example, what is F_7? Is it $W_1 + W_2$ or $W_1 + W_2 + W_3$; or is F_2 $W_1 + W_2$ or W_1 only? These questions are not easily answered, but when we write F_{27}, then that is plainly $W_1 + W_2$, because it is for any point between 2 and 7. (See N girder, page 326.)]

These two formulæ give us the numerical value of the stress in any bar, and for its nature we study the distortion as already explained.

We will now show the application of these formulæ to certain definite cases.

Case I.—Warren girder supported at the ends, loaded at any one intermediate joint. The figure shows

a girder of 5 divisions in the lower boom, loaded at the second joint from the left.

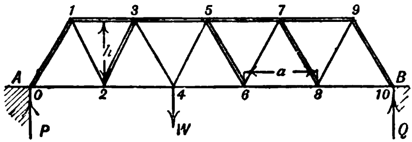


Fig. 232.

Number the joints as shown, and let the length of each division be a. Then

$$h = \frac{a}{2} \cdot \tan 60^\circ = \frac{a\sqrt{3}}{2}.$$

First, we find P and Q, the supporting forces.

Taking moments about A,

$$Q \times 5a = W \times 2a, \quad \therefore \ Q = \tfrac{2}{5}W,$$

and

$$\therefore \ P = \tfrac{3}{5}W.$$

Stress in Booms.—We will commence with the booms, and take the bars in order of the joints to which they are opposite, and we may say in commencing that much time is saved in these questions by working systematically. Then

$$H_{02} \cdot h = M_1 = P \cdot \frac{a}{2},$$
$$H_{13} \cdot h = M_2 = P \cdot a,$$
$$H_{24} \cdot h = M_3 = P \cdot \frac{3a}{2},$$
$$H_{35} \cdot h = M_4 = P \cdot 2a.$$

Now we will alter our procedure, since both P and W will lie on the left, so we look to the right. We will not, however, proceed farther with this mode of writing down the results, because it is best done as a part of a tabular method of calculation, by which greater speed and accuracy are obtained. Make out then a table as follows :—

Joint	1	2	3	4	5	6	7	8	9
Bar	0 2	1 3	2 4	3 5	4 6	5 7	6 8	7 9	8 10
M	$P.\frac{a}{2}$	Pa	$P\frac{3a}{2}$	$P.2a$	$Q.\frac{5a}{2}$	$Q.2a$	$Q.\frac{3a}{2}$	$Q.a$	$Q.\frac{a}{2}$
$H=\frac{2M}{a\sqrt{3}}$	$\frac{P}{\sqrt{3}}$	$\frac{2P}{\sqrt{3}}$	$\frac{3P}{\sqrt{3}}$	$\frac{4P}{\sqrt{3}}$	$\frac{5Q}{\sqrt{3}}$	$\frac{4Q}{\sqrt{3}}$	$\frac{3Q}{\sqrt{3}}$	$\frac{2Q}{\sqrt{3}}$	$\frac{Q}{\sqrt{3}}$

In the top line we write the joints in order, next the opposite bars, and by adopting the method of numbering of Fig. 232, *i.e.* running along the diagonals, we can quickly write down the bars, since the three numbers are in order thus $\overset{2}{1\ \ 3}$.

Next, write down the values of M; and this we can quickly do, since they are in arithmetic progression from each end, only being careful to stop in each case at the loaded joint.

Then the last line comes at once by multiplication by 2, and division by $a\sqrt{3}$, for

$$H=\frac{M}{h}=\frac{M}{\frac{a\sqrt{3}}{2}}=\frac{2}{a\sqrt{3}}M.$$

Then in any practical case the values of P and Q must be inserted and the square root taken out, the final results should not be left with surds in them.

Now, for the nature of the stresses, we see that if any

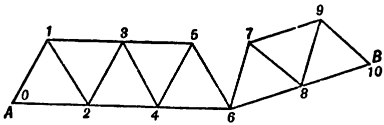


Fig. 233.

top bar, say 57, be removed, the joints approach as Fig.

233. Hence all the top bars are in compression, and similarly all the bottom bars are in tension.

Struts and Ties.—One way of denoting the stress in a bar is, as we have explained, by arrows; but this becomes complex when there are a large number of bars, and hence a simpler system is adopted, viz. by double lining all the bars in compression, as Fig. 232. Bars in compression are also distinguished from those in tension by a distinctive name, being known as **Struts**, while tension bars are called **Ties.**

Stress in Diagonals.—Examination shows that all the diagonals leaning towards the weight are struts, and those leaning away from it ties. This we have shown by lining the struts.

For the magnitudes of the stresses we have

$$S = F \sec\theta = \frac{2F}{\sqrt{3}}.$$

The calculation is simple, for from 0 to 4 F is P, and from 4 to 10 F is Q; we could then say all diagonals to left of load are stressed to $2P/\sqrt{3}$, and those to the right to $2Q/\sqrt{3}$.

It is better, however, for a reason we shall soon see, to tabulate them; and also we will in the table denote the nature of the stress, which we have not before done, using + for tension and − for compression. We get then the following table:—

Bar	0.1	1.2	2.3	3.4	4.5	5.6	6.7	7.8	8.9	9.10
F	P	P	P	P	Q	Q	Q	Q	Q	Q
$S=\frac{2F}{\sqrt{3}}$	$-\frac{2P}{\sqrt{3}}$	$+\frac{2P}{\sqrt{3}}$	$-\frac{2P}{\sqrt{3}}$	$+\frac{2P}{\sqrt{3}}$	$+\frac{2Q}{\sqrt{3}}$	$-\frac{2Q}{\sqrt{3}}$	$+\frac{2Q}{\sqrt{3}}$	$-\frac{2Q}{\sqrt{3}}$	$+\frac{2Q}{\sqrt{3}}$	$-\frac{2Q}{\sqrt{3}}$

Suppose now we consider the effect which would be produced by loading another joint, in addition to 4.

Then, for the booms, this would produce compression in the top and tension in the bottom boom, whichever joint be loaded, and the effect would be an increase of stress in all the bars of the booms.

For the diagonals, however, we cannot say this, because the effect of the new load will be to produce tension, or to tend to produce it, in some of the bars already in compression, and *vice versâ*—*e.g.* let the new load be at 6, then it would produce tension in 56, but there is already compression in 56, so that we do not know whether the resultant effect will be tension or compression, or perhaps even no stress at all, if the tension which the load at 6 would produce alone should be equal to the compression produced by the load at 4.

We have here been using the principle of Superposition, and this principle is well suited for application to the case in which a girder is loaded at more than one joint, because of the ease with which by the tabular method the separate effects of each load can be found. As an example we will consider

Case II.—A Warren girder, supported at the ends, carrying a uniformly loaded platform. First, to find the load at each joint (page 312). Let

$2w$ = load per ft. run of platform in lbs.

Then, if as usual there be two girders, each one supports a platform loaded with w lbs. per foot run.

Take now any three consecutive joints of the boom on which the load rests. In this case we take the bottom boom. Let A, B, and C (Fig. 234) represent in skeleton the joints, and A′B′C′ the platform resting on the cross girders.

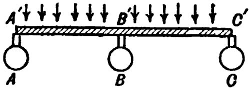


Fig. 234.

Then the piece of platform A′B′ is supported at A′ and B′

and loaded uniformly, so that half of the load on it is carried by B. Similarly half the load on B′C′ is carried by B,

$$\therefore \text{ Load on B} = \frac{w.\text{A}'\text{B}' + w.\text{B}'\text{C}'}{2} = \frac{w.\text{A}'\text{C}'}{2},$$

and we see that we have the general principle that *the load on any joint is one half the load on the platform between the two joints on either side of it.* If the joint be an end joint, then of course there is platform on one side only, and it carries half the load on the piece between itself and the next joint, *e.g.* if A be an end joint,

$$\text{Load at A} = \tfrac{1}{2}w.\text{A}'\text{B}'.$$

Applying this to our case, the girder is loaded as in Fig. 235, and we must now dismiss all thought of dis-

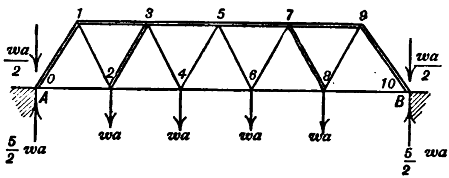


Fig. 235.

tributed loading from our minds, because the load on the girder is *not* a distributed but a concentrated one, *i.e.* concentrated at the several joints, not of course at one point only.

Loads at Points of Support.—A question must now be considered which often causes trouble.

At 0 and 10 there are loads $wa/2$, $wa/2$. Now it is evident that no amount of loading at 0 and 10 can produce any stress in the bars of the girder, because 0 and 10 rest directly on the ground, and putting a load on top of the joint simply squeezes it between the load and the ground. Since these loads produce no effect we may, if we please, neglect them. But if we neglect them we

must do so entirely, and treat the question as if they did not exist; it is, in fact, safer to draw a fresh figure as Fig. 236.

The point where mistakes arise is in finding the supporting forces. Looking at Fig. 236, these are $2wa$, $2wa$. But the total load on the bridge is not $4wa$ but $5wa$, and hence the supporting force at each end must be $5wa/2$; how then can it be correct to take $2wa$? The reason is, that although $5wa/2$ must be the upward pressure of the ground, yet there is also the $wa/2$ at 0 and 10, and thus the *resultant* force at 0 is $2wa$. It does not matter, then, which way we treat the question, the results will be the same, but what we must not

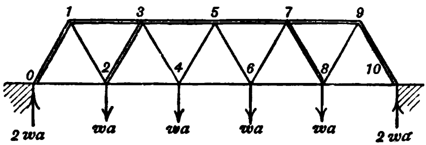


Fig. 236

do is to take $5wa/2$ as the supporting force, and then neglect the $wa/2$. To avoid confusion $5wa/2$ may be called the Real, and $2wa$ the Virtual, supporting force.

The preceding work will not, however, affect our method of proceeding, because we are going to treat each load separately, and shall not require the total supporting force, but it comes naturally after we consider loading at the joints, and hence we have here put it in.

We will now then take each weight in turn, commencing on the left, and of course omitting the $wa/2$, its effect being *nil*. We use the tabular method as below, taking first the

Booms :—

Joint	1	2	3	4	5	6	7	8	9
Bar	0 2	1 3	2 4	3 5	4 6	5 7	6 8	7 9	8 10
Load at 2, $P=\frac{4}{5}wa$, $Q=\frac{1}{5}wa$ — M	$\frac{4}{10}wa^2$	$\frac{8}{10}wa^2$	$\frac{7}{10}wa^2$	$\frac{6}{10}wa^2$	$\frac{5}{10}wa^2$	$\frac{4}{10}wa^2$	$\frac{3}{10}wa^2$	$\frac{2}{10}wa^2$	$\frac{1}{10}wa^2$
H	$\frac{4wa}{5\sqrt{3}}$	$\frac{8wa}{5\sqrt{3}}$	$\frac{7wa}{5\sqrt{3}}$	$\frac{6wa}{5\sqrt{3}}$	$\frac{5wa}{5\sqrt{3}}$	$\frac{4wa}{5\sqrt{3}}$	$\frac{3wa}{5\sqrt{3}}$	$\frac{2wa}{5\sqrt{3}}$	$\frac{wa}{5\sqrt{3}}$
Load at 4, $P=\frac{3}{5}wa$, $Q=\frac{2}{5}wa$ — M	$\frac{3}{10}wa^2$	$\frac{6}{10}wa^2$	$\frac{9}{10}wa^2$	$\frac{12}{10}wa^2$	$\frac{10}{10}wa^2$	$\frac{8}{10}wa^2$	$\frac{6}{10}wa^2$	$\frac{4}{10}wa^2$	$\frac{2}{10}wa^2$
H	$\frac{3wa}{5\sqrt{3}}$	$\frac{6wa}{5\sqrt{3}}$	$\frac{9wa}{5\sqrt{3}}$	$\frac{12wa}{5\sqrt{3}}$	$\frac{10wa}{5\sqrt{3}}$	$\frac{8wa}{5\sqrt{3}}$	$\frac{6wa}{5\sqrt{3}}$	$\frac{4wa}{5\sqrt{3}}$	$\frac{2wa}{5\sqrt{3}}$
Load at 6 – H	$\frac{2wa}{5\sqrt{3}}$	$\frac{4wa}{5\sqrt{3}}$	$\frac{6wa}{5\sqrt{3}}$	$\frac{8wa}{5\sqrt{3}}$	$\frac{10wa}{5\sqrt{3}}$	$\frac{12wa}{5\sqrt{3}}$	$\frac{9wa}{5\sqrt{3}}$	$\frac{6wa}{5\sqrt{3}}$	$\frac{3wa}{5\sqrt{3}}$
Load at 8 – H	$\frac{1wa}{5\sqrt{3}}$	$\frac{2wa}{5\sqrt{3}}$	$\frac{3wa}{5\sqrt{3}}$	$\frac{4wa}{5\sqrt{3}}$	$\frac{5wa}{5\sqrt{3}}$	$\frac{6wa}{5\sqrt{3}}$	$\frac{7wa}{5\sqrt{3}}$	$\frac{8wa}{5\sqrt{3}}$	$\frac{4wa}{5\sqrt{3}}$
$\therefore$ Total $H=\frac{wa}{5\sqrt{3}}\times$	10	20	25	30	30	30	25	20	10

The process appears long, but is in reality very quickly done, as after the first rows there is no need to put in the wa^2, wa, or $5\sqrt{3}$ when working an actual example. Also for 6 and 8 we do not require M, because 6 and 4 being symmetrically situated the line of H's for 6 is simply 4 turned end for end; the same applies to 8 and 2, as can in each case be seen from the table.

Diagonals.—We write the table as follows :—

		Bar	0.1	1.2	2.3	3.4	4.5	5.6	6.7	7.8	8.9	9.10
Load at 2	$F=wa\times$		$\frac{4}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
	$S=\frac{wa}{5\sqrt{3}}\times$		-8	$+8$	$+2$	-2	$+2$	-2	$+2$	-2	$+2$	-2
Load at 4	$F=wa\times$		$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
	$S=\frac{wa}{5\sqrt{3}}\times$		-6	$+6$	-6	$+6$	$+4$	-4	$+4$	-4	$+4$	-4
Load at 6	$S=\frac{wa}{5\sqrt{3}}\times$		-4	$+4$	4	$+4$	-4	$+4$	$+6$	-6	$+6$	-6
Load at 8	$S=\frac{wa}{5\sqrt{3}}\times$		-2	$+2$	-2	$+2$	-2	$+2$	-2	$+2$	$+8$	-8
Total	$S=\frac{wa}{5\sqrt{3}}\times$		-20	$+20$	-10	$+10$	0	0	$+10$	-10	$+20$	-20

6 and 8 are again 4 and 2 reversed. The sign of F is not put in, because for the sign of S we look not to F but to the figure.

The case can also be treated directly from the general formulæ, and the student should in this way verify the results we have obtained by superposition.

The N Girder.—The general treatment is identical with that of the Warren type, but there are one or two points which require examination. The figure shows a

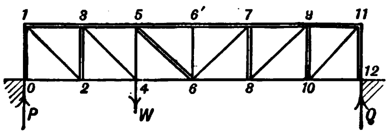


Fig. 237.

girder in six divisions, the sloping bars leaning away from the centre, as is always the case, and inclined, as is usual, at 45°. Then for the diagonals,

$$S=F\sec 45^\circ=F\sqrt{2},$$

and for the uprights

$$S = F \sec 0° = F.$$

Let now the girder be loaded at, say, joint 4 of the lower boom. Then

$$P = \tfrac{3}{8}W, \quad Q = \tfrac{5}{8}W.$$

Now the question is, What is S_{45}? 45 being directly over W, F there changes from P to Q; but S_{45} must have a definite value, and so we have to inquire what value of F is there to be taken.

We shall now have to examine 45 from first principles. Its duty is to keep up the end 4 of the body

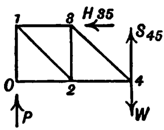


Fig. 238.

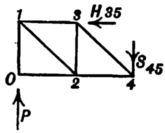


Fig. 239.

0134 (Fig. 238). This body is acted on by P, W, H_{35}, and S_{45}. Resolving vertically

$$S_{45} + P = W,$$
$$\therefore S_{45} = W - P = Q.$$

So it appears that F is to be taken as Q, and it is tension.

If, however, W had been at the top joint 5, then we should have simply

$$S_{45} = P,$$

from the equilibrium of 0134, as before (Fig. 239), and S would be compression.

This special examination must be applied to all vertical bars, which are directly under or over loaded joints.

There is another bar which requires special consideration, viz. 66′.

If we remove 66′, then without there be a load at 6′, no effect at all is produced, for in theory the bar 57 can withstand either pull or thrust, even though it be jointed at 6′. Practically, however, we know that a jointed bar cannot take compression, because some slight accidental circumstance bends the joint a little, and then the strength is gone. This, then, is the reason for fitting 66′, that is, to keep 57 straight, and so enable it to take thrust; and then 57 being straight, there can, from the equilibrium of the joint at 6′, be no stress in 66′ (see page 439). In any case we cannot calculate what stress is on 66′, because it depends on accidental causes; but we know it will, in the absence of a load at 6′, be small. If there be a load at 6′ then 66′ must wholly support it, because the jointed bar 57 has no sideways strength (see page 439).

EXAMPLES.

1. A Warren girder 9 ft. long on top projects from a wall, the top boom being in three divisions. A load of two tons is placed at the end. Find the stresses in all the bars.

Ans. Number as in Fig. 223, then the stresses are, in tons—

Top boom	12		23		34		Bar.
	1.16		3.47		5.77		Stress.
Bottom boom	56		67		78		Bar.
	−2.31		−4·62		−6.94		Stress.
Diagonals	15	52	26	63	37	74	Bar.
	−2.31	+2.31	−2.31	+2.31	−2.31	+2.31	Stress.

2. Obtain the above results when either of the other joints of the top boom is loaded, and hence deduce the result when a distributed load of $\frac{2}{3}$ tons per foot run is carried.

Ans.

2 tons at 2.

Top boom	12		23		34		Bar.
	0		1.16		3.47		Stress.
Bottom boom	56		67		78		Bar.
	0		−2.31		−4.62		Stress.
Diagonals	15	52	26	63	37	74	Bar.
	0	0	−2.31	+2.31	−2.31	+2.31	Stress.

2 tons at 3.

Top boom	12		23		34		Bar.
	0		0		1.16		Stress.
Bottom boom	56		67		78		Bar.
	0		0		−2.31		Stress.
Diagonals	15	52	26	63	37	74	Bar.
	0	0	0	0	+2.31	+2.31	Stress.

The load at 1 will be only 1 ton, therefore halve the first results and add the last two for the distributed load.

3. A Linville girder in 8 divisions carries a load of 25 tons at the joint next on the left to the centre of the lower boom. Find the stresses in all the bars.

Ans.

Booms.				Web.			
Bar.	Stress.	Bar.	Stress.	Bar.	Stress.	Bar.	Stress.
1.3	−15	0.2	0	0.1	−15	1.2	+21.2
3.5	−30	2.4	15	2.3	−15	3.4	+21.2
5.7	−45	4.6	30	4.5	−15	5.6	+21.2
7.8′ / 8′.9	−40	6.8	45	6.7	+10	7.8	−14.1
		8.10	30	8.8′	0	8.9	+14.1
9.11	−30	10.12	20	9.10	−10	10.11	+14.1
11.13	−20	12.14	10	11.12	−10	12.13	+14.1
13.15	−10	14.16	0	13.14	−10	14.15	+14.1
				15.16	−10		

4. What would be the effect on the results of the preceding if the load were placed at the top joint?

Ans. The stress in 6.7 would be 15 tons tension.

5. A bridge 100 feet span is supported by a pair of Warren girders under the platform, which rests on the joints of the upper boom, which is in 12 divisions, and also, by struts, on the joints of the lower boom. The bridge is loaded with $1\frac{1}{4}$ tons per ft. run. Find the stresses on the bars.

Ans. 23 joints are loaded with $2\frac{29}{48}$ tons at each, and each end joint with $1\frac{29}{96}$ tons. The virtual supporting forces are, therefore, 30 tons almost exactly; and apply the tabular method.

NOTE.—The truss described in this chapter as an N girder is also known as a Linville, a Pratt, or a Whipple-Murphy girder or truss, after various designers, who have made small changes in details of construction.

CHAPTER XVII

LATTICE GIRDER—TRAVELLING LOAD—COUNTERBRACING

AT the end of the last chapter we saw that there was in the N girder one bar, viz. the central upright, on which the stress was theoretically zero; but which was nevertheless practically necessary, since without it the least jar would destroy the structure. If we inquire further we shall find two more bars which are in this condition, viz. the end bars of the lower boom.

For in theory the girder could stand as Fig. 240, but

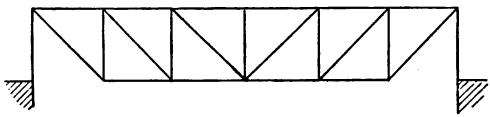


Fig. 240.

its equilibrium would be unstable, and the least shake would overturn it.

Redundant Bars—Lattice Girder.—Bars such as we have considered, not being in strict theory necessary, may be called redundant, but in the cases we have so far considered they are redundant only for particular kinds of loading, *e.g.* the centre upright is not redundant if there be a load at its top joint, and the end bars of the booms are not redundant if there be any force not exactly vertical. These bars, then, are necessary if the

girder is required to keep its shape under all kinds of loading, and hence the term redundant is not applied to them.

But now consiaer the common type of girder shown in Fig. 241, and known as the lattice or trellis girder.

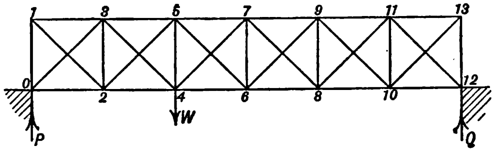


Fig. 241.

Here in each space between two uprights, or bay, two diagonals are fitted, whereas one is quite sufficient to prevent any change of form except actual breaking. In this case the extra diagonal is truly redundant, and the structure is called a redundant structure.

These structures are common, for reasons explained later in the chapter, hence we will examine this case to see how the stresses can be determined; and the method pursued will be in principle applicable to all cases of redundancy.

First, we shall find that our former rules fail, for—

Number the joints, and suppose the girder be loaded with W at 4 (Fig. 241). Let P and Q be the supporting forces. We shall now find it impossible to obtain the stress in any bar by considering the equilibrium of a portion of the girder. For example take 0132.

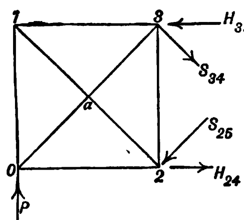


Fig. 242.

Then we have acting P known, and four forces, H_{35}, H_{24}, S_{25} and S_{34} unknown. We have drawn the H's as shown, since it is evident that all top bars are

compressed and bottom bars extended. The direction of the S's can also be seen by imagining both 34 and 25 removed, then the girder distorts as Fig. 243, 34 lengthening and 25 shortening, hence 34 is in tension and 25 in compression. It is not necessary, however, to spend

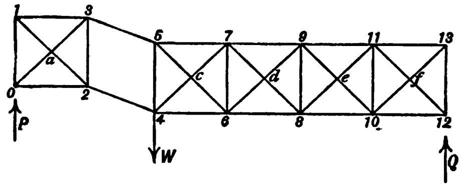


Fig. 243.

time over this, since if a force be put in the wrong direction no harm will result, but the answer obtained will be negative, and then the arrow must be reversed.

Now we cannot, either by resolving or by taking moments, obtain a definite result for any of these forces. For vertically,

$$P = S_{34}\cos\theta + S_{25}\cos\theta \qquad (1).$$

(θ = inclination of bars to vertical.)

Horizontally

$$H_{35} + S_{25}\sin\theta = H_{24} + S_{34}\sin\theta \qquad (2).$$

We can take moments about any point, either in or out of the body, and the best point to select is the intersection of the S's, that is (b) in Fig. 244, since the moments of the S's then vanish, and we have

$$H_{35}\frac{h}{2} + H_{24}\frac{h}{2} = \text{moment of P about } b,$$
$$= M_b \qquad (3),$$

where h = height of girder.

These equations then give

$$S_{34} + S_{25} = P\sec\theta \quad (\text{from } (1)),$$
$$= F\sec\theta.$$

Since $F=P$, and it can be easily seen that this holds generally. For consider 1760, then we obtain

$$S_{69}+S_{78}=(W-P)\sec\theta=F\sec\theta,$$

and similarly if there were any number of loads.

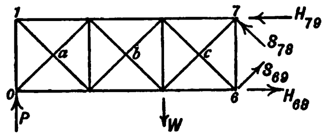


Fig. 244.

Next (2) gives,

$$H_{35}-H_{24}=(S_{34}-S_{25})\sin\theta,$$

and (3)

$$(H_{35}+H_{24})\frac{h}{2}=M_b.$$

This latter can be shown also to hold generally by considering 1760 (Fig. 244).

These equations are similar in form to $S=F\sec\theta$, $Hh=M$; but we have two forces S, and two forces H, connected by one more equation (2), but this is insufficient to determine them.

The solutions then are not in themselves determinate, and we can only get definite results by obtaining some fresh relation between the forces; and this relation must come from practical considerations, because we have got all we can from theory.

We will commence now to examine further, and first notice that taking the general equation

$$(H+H')\frac{h}{2}=M.$$

H and H′ being the stresses in two corresponding bars of the respective booms, M is taken about the intersection of the diagonals. Now the diagonals would always be continuous bars, often connected to each other

by riveting, and in rare instances by a pin joint. We shall, however, treat them as if they were actually jointed at *a*, *b*, *c*, etc.; those at *b*, for instance, being taken as four separate bars, 3*b*, 2*b*, 5*b*, 4*b*; and we can now call *b* the joint opposite to 24 or 35, so M is still the B. M. at the opposite joint.

[We must notice that the assumption here made is perfectly legitimate, because 34 being in tension would be just as strong jointed as not, while although jointing 25 may be supposed to impair its power of resisting compression, yet this is not so, because 3*b* and 4*b* are there to keep it in a straight line, and it is then just as strong jointed as solid.]

We are not, however, any nearer the determination of the forces, although the work can in many cases be simplified by treating *a*, *b*, etc., as joints, so we must examine farther still.

We shall have to consider a question which hitherto has been neglected, viz. the material of which the girder is composed—and we shall find that on this depends the question as to the extra relation between the forces; also we shall see why these extra or redundant diagonals, or braces, are fitted.

Consider the bay 2345 of the girder. Then 2345 tends to change shape from (*a*) to (*b*). Now this change of shape can be resisted either by fitting a diagonal 34

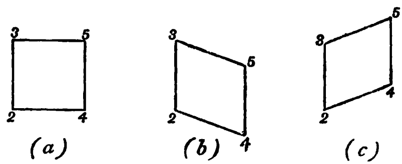


Fig. 245.

to take tension or 25 to take compression, and so far as we have seen at present it is quite immaterial which we fit. But now suppose the structure is of wrought-iron

or steel, and that F, the shearing force, which causes the distortion, is not very large; then we should at once decide to fit 34 and not 25; because 34 need be only a slender rod, while across 25 a thin rod would be of no use, since a very little compression would bend it sideways, and we should have to increase its dimensions considerably to prevent this.

If, however, the stress were considerable, and the bays short, so that the transverse dimensions of the brace are necessarily large, while its length is moderate, then it would be immaterial which we fitted. In this case the brace is usually a flat bar.

Take, however, now the case in which wood is the material. Then in the first place the transverse dimensions would generally be considerable, but besides this another question, viz. fastenings, comes in. In this case, not only is there no preference for fitting 34, but there is a great objection to it on account of the great difficulty of making a timber joint which can take tension; in metals there is no such difficulty. We should then fit a brace across 25, and it would take the compression by its ends butting against the joints, fastening being required only to keep it in place.

Now proceed further to consider what would be necessary if the girder were so loaded that 2345 tended to change shape to (*c*) and not (*b*), *i.e.* suppose the shearing force to be changed from + to −. Then all we have said above must be reversed: if wrought-iron a slender rod would be across 25, if wood a strut across 34.

Combining the preceding we now see that if the bridge or girder be sometimes loaded so that the shear over the division 2345 is plus, and at other times so loaded that it is minus, then we should fit either two slender ties 25 and 34, or two wooden struts; and it would be better to do this as a rule than to make, say, 25, if of wrought-iron, thick enough to be able to resist

the change of shape in Fig. 245, or if of wood to attempt to make a tension joint.

In the second case of wrought-iron, viz. the flat bar stay, one would be sufficient in many cases, but even here it is usual to fit two; and, as before stated, in all these cases the braces are usually connected where they cross for the sake of stiffness.

We shall see at the end of the chapter that bridges are subjected to loads which produce the change of shear spoken of above, and hence we see why these redundant bars are fitted; and we shall now see also that the same considerations which show us why they are fitted also assist us in determining the stresses on them.

Consider the first case in wrought-iron, then 25 and 34 are slender rods, practically useless against compression; hence when a tendency to change, as Fig. 245 (*a*), comes on the bay, 25, instead of resisting the change equally with 34, gives way by bending, directly it is in the least compressed, and so the whole work of resisting the change is practically thrown on 34. Hence, then, we say that—

When the braces are long and slender, the question is to be treated as if only the one in tension were fitted.

Evidently when we consider the case of wood, exactly the opposite is true, since the fastenings of the ends of 34 will be loose enough to admit of an extension of the distance 34, without any stress being produced in the piece 34, hence 25 does the whole duty, and therefore—

In wooden structures, consider only the brace in compression.

Next, consider the intermediate case of the flat bar braces; then these are equally capable of resisting extension or compression, hence since 34 lengthens and 25 shortens practically equal amounts, we divide the stress equally between them, and hence—

In metal structures, where the braces are not slender

compared with their lengths, divide the stress equally between the two.

This latter is the most common in parallel bridge girders—the first case in metal structures applying chiefly to roof trusses—so we shall treat our parallel lattice girder in this way.

Before proceeding, however, the student should notice that the determination of the stresses depends on questions of material, and also of workmanship as regards fastenings, etc., which did not enter into our original equations at all. We make certain suppositions of which we have given the three principal, but it is easily seen that there will be many cases in which intermediate suppositions must be made, *e.g.* the bar 25 may offer a decided resistance to compression, but yet not quite so much as 34 does to extension, then we should make S_{25} say a half of S_{34} instead of equal to it—this necessity for consideration of other than purely statical questions constitutes the great distinction between redundant and non-redundant structures.

Proceeding now with our lattice girder, we had as general equations, for the four bars of any bay,

$$(H+H')\frac{h}{2}=M \text{ (about middle joint of bay)} \qquad (1),$$

$$S+S'=F\sec\theta \qquad (2),$$

$$(S-S')\sin\theta=H-H' \qquad (3),$$

S and S′ being the stresses in the diagonals, and H, H′ those in the booms. These equations are simply those of pages 232 and 233, written in a more general form.

We are now going to *assume*

$$S=S',$$

whence it at once follows that

$$H=H' \text{ (from (3))},$$

and therefore

$$2H\cdot\frac{h}{2}=M, \quad \text{or } Hh=M,$$

where H is the total stress in *either* boom bar, not in *both* (a common mistake). Also

$$2S=F\sec\theta, \quad \therefore S=\frac{F}{2}\sec\theta.$$

These then are the general equations for all cases in which the assumption holds good, and we will now apply them.

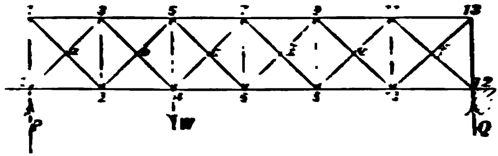


Fig. 246.

The figure (Fig. 241) on page 331 is here reproduced (Fig. 246). θ is $\pi/4$, and let a be the length and height of each bay. Then first

$$P=\frac{2W}{3}, \quad Q=\frac{W}{3}.$$

Now proceeding,

$$H_{13}=H_{02}=\frac{M_a}{h},$$

$$=\frac{Pa}{2}\div a=\frac{W}{3};$$

$$H_{35}=H_{24}=\frac{M_b}{h},$$

$$=P\cdot\frac{3a}{2}\div a=W;$$

$$H_{57}=H_{46}=\frac{M_c}{h},$$

$$=Q\cdot\frac{7a}{2}\div a=\tfrac{7}{6}W;$$

$$H_{79}=H_{68}=\frac{M_d}{h},$$

$$=Q\cdot\frac{5a}{2}\div a=\tfrac{5}{6}W;$$

and then

$$H_{9.11} = H_{8.10} = \tfrac{2}{3} W,$$
$$H_{11.13} = H_{10.12} = \tfrac{1}{3} W,$$

since the M's are plainly in arithmetic progression, whence so also are the H's. For the diagonals

$$S_{12} = S_{03} = S_{34} = S_{25} = \frac{P}{2} \sec \theta = \frac{W\sqrt{2}}{3},$$

since F = P anywhere between 0 and 4; and for all the diagonals to the right of W,

$$S = \frac{Q}{2} \sec \theta = \frac{W\sqrt{2}}{6},$$

We have now the uprights left, and they must be considered separately.

First 01; this has to keep up the end of 21, being assisted in doing so by 31, and being vertical while 31 is horizontal, 01 takes the vertical component of S_{21} while 31 takes the horizonal component,

$$\therefore S_{01} = S_{21} \cos \theta = \frac{W}{3},$$

and θ being $\pi/4$ this will also be the value of H_{31}, which we have already found is so.

Next 23 keeps up the ends of 03 and 34, but 03 pushes 3 up as much as 34 pulls it down, since $S_{03} = S_{34}$, hence there is no total effect, and $S_{23} = 0$, so that 23 is *for this loading* not needed (compare 66′, page 327).

Reasoning in the same way we find

$$S_{67} = S_{89} = S_{910} = 0,$$

and

$$S_{12.13} = S_{10.13} \cos \theta = \frac{W}{6}.$$

We have left now 45 directly over W. This keeps down the joint 5, which is pushed up by 65 and also by 25; 35 and 57 being horizontal can neither push up nor down. Hence

$$S_{45} = S_{65} \cos \theta + S_{25} \cos \theta,$$
$$= \frac{W}{6} + \frac{W}{3} = \frac{W}{2},$$

and it pulls 5 down, so it is in tension.

[This we should have expected, since at 4 it helps to support W, but it does not from that necessarily follow that it is in tension, and we must in all cases make sure by actual calculation.]

We have now found all the stresses due to a load at one joint, and by the same method we can obtain them when any other joint or number of joints are loaded.

Bridge Girder Loads—Permanent and Travelling.—We have shown that the reason why it was necessary to have two diagonals in one bay was to enable the bay to withstand either + or − shear. This process of doubling the diagonal is called Counterbracing, and we are now about to inquire how it is that the shear on some bays is sometimes +, and at other times −, and so renders counterbracing necessary. For this purpose we must consider what loads a bridge has to bear.

The first, and in long spans most important, load, is the weight of the bridge itself, and of the platforms, rails, etc., which it carries. This load is always on the bridge, and is hence called the **Permanent** or **Dead Load.**

Secondly, there are the loads due to the passage of bodies across the bridge. These loads, being continually coming on and going off again, are called **Live Loads** or **Travelling Loads.**

The permanent load will be in all cases a distributed load, and for parallel girders may be taken as uniformly distributed. It is then equivalent to a set of equal loads applied at the joints of whichever boom the platform rests on. We can then in a girder so place the diagonals that under the permanent load they shall be all in tension.

Fig. 247 shows the proper arrangement; each joint is loaded with W, say, and there is one diagonal in each bay, which we can easily see is in tension, for—

There is symmetry, and $S_{66'} = 0$ (page 326); therefore 56 and 67 are in tension, since they support W.

Then it follows that 45 and 78 are in compression,

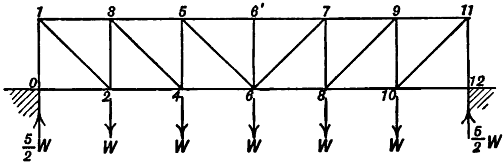


Fig. 247.

since they hold up the ends of 56 and 67 respectively; and hence 43 and 89 are in tension, and so on.

[We see here that although the diagonals are in tension, yet the uprights are in compression, and it may be asked, What is the gain? It is this: that the uprights are shorter than the diagonals—in some cases considerably so—and are hence better suited to withstand compression, since they bend less. As a general principle we try to keep short bars in compression and long bars in tension.]

Effect of Travelling Load.—We now consider the effect produced when a load, such as a locomotive,

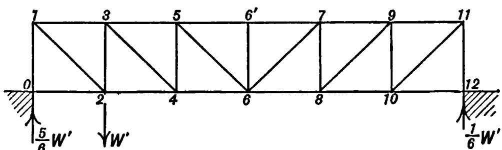


Fig. 248.

crosses the bridge. Let the weight of the locomotive be W′, and we will treat it as a concentrated load, examining its effect as it reaches each joint in order. Suppose it to come on from the left. Then (Fig. 247) we have, before it comes on,

$$\left.\begin{array}{l} S_{1.2} = S_{10.11} = \frac{5}{2}\,W \sec\theta \\ S_{3.4} = S_{8.9} \;\;= \frac{3}{2}\,W \sec\theta \\ S_{56} \;= S_{67} \;\;\;\, = \frac{1}{2}\,W \sec\theta \end{array}\right\}\text{all tension.}$$

Let W′ come to 2. Then *due to W′* (Fig. 248),

$$S_{1.2} = \tfrac{5}{6}\,W' \sec\theta \text{ tension,}$$
$$S_{10.11} = S_{8.9} = S_{67} = \tfrac{1}{6}\,W' \sec\theta \text{ tension.}$$

But

$$S_{56} = S_{34} = \tfrac{1}{6}\,W' \sec\theta \text{ compression,}$$

because they slope the wrong way for tension (see Fig. 237, page 325).

Now 34 had originally $\frac{3}{2}$ W sec θ tension, and hence if

$$\tfrac{1}{6}\,W' > \tfrac{3}{2}\,W,$$

the total effect when W′ is at 2 is that 34 is in compression, and

$$S_{34} = (\tfrac{1}{6}\,W' - \tfrac{3}{2}\,W) \text{ compression.}$$

Therefore 56 is also in compression, for

$$S_{56} = (\tfrac{1}{6}\,W' - \tfrac{1}{2}\,W) \sec\theta \text{ compression,}$$

and if $\frac{1}{6}\,W' - \frac{3}{2}\,W$ is plus, much more will $\frac{1}{6}\,W' - \frac{1}{2}\,W$ be plus.

But 56 will be in compression if $\frac{1}{6}\,W'$ be not greater than $\frac{3}{2}\,W$, so long as it is greater than $\frac{1}{2}\,W$. If then

$$\tfrac{1}{6}\,W' > \tfrac{3}{2}\,W,$$

and we do not desire the braces to take compression, we must counterbrace both bays, 2354 and 456′6. If, however,

$$\tfrac{1}{6}\,W' < \tfrac{3}{2}\,W \text{ but} > \tfrac{1}{2}\,W,$$

we need, so far as we can see at present, only counterbrace 456′6.

The effect of W′ at 2 on the bars 1.2, 6.7, 8.9, and 10.11 is to increase the tension in them; but this does not, for our present purpose, concern us, because we counterbrace not to decrease tension, but to prevent compression. Suppose finally that

$$\tfrac{1}{6}\,W' < \tfrac{1}{2}\,W,$$

then the stresses in 23 and 56 are reduced to

$$S_{23} = (\tfrac{3}{2}\,W - \tfrac{1}{6}\,W')\sec\theta,$$
$$S_{56} = (\tfrac{1}{2}\,W - \tfrac{1}{6}\,W')\sec\theta,$$

but they remain tensions, and thus so far no counter-bracing at all would be required.

Next, let W′ move to 4. Then due to W′ at 4.

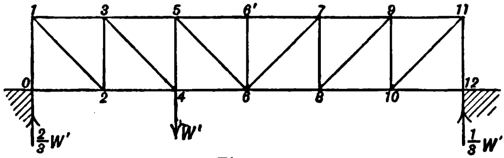


Fig. 249.

$$S_{12} = S_{34} = \tfrac{2}{3}\,W'\sec\theta \text{ tension},$$
$$S_{10.11} = S_{8.9} = S_{67} = \tfrac{1}{3}\,W'\sec\theta \text{ tension}.$$

And we have only 56 with compression due to W′, its amount being

$$S_{56} = \tfrac{1}{3}\,W'\sec\theta.$$

The total stress then in 56 is, total,

$$S_{56} = (\tfrac{1}{2}\,W - \tfrac{1}{3}\,W')\sec\theta \text{ tension},$$

or

$$(\tfrac{1}{3}\,W' - \tfrac{1}{2}\,W)\sec\theta \text{ compression},$$

these of course being identical. If then

$$\tfrac{1}{3}\,W' > \tfrac{1}{2}\,W,$$

there would be compression in 56, and we should counterbrace. All the other bars are in tension, so only this one bay needs counterbracing. If

$$\tfrac{1}{3}\,W' < \tfrac{1}{2}\,W,$$

no counterbracing at all is required.

Next, let W′ come to 6.

Then the only effect is to increase the tension in all the diagonals, hence no counterbracing is required for this position of the travelling load.

We have now traced the load half-way across, and we will summarise our results as follows :—

When W′ is at 2 it produces compression in 34, but not when at 4 or 6. When W′ is at 2 or 4 it produces compression in 56, but its effect is greatest when it is at 4 directly under the end of 56, being then $\frac{1}{3}$ W′ sec θ against $\frac{1}{6}$ W′ sec θ when at 2.

The question whether 34 requires a counterbrace is then settled when W′ is at 2, while for 56 it may be settled at 2 but must be settled at 4. If

$$\tfrac{1}{6}\,W' > \tfrac{3}{2}\,W, \quad \text{or } W' > 9\,W,$$

doth 34 and 56 require counterbraces ; while if

$$\tfrac{1}{3}\,W' > \tfrac{1}{2}\,W, \text{ or } W' > \tfrac{3}{2}\,W \text{ but } < 9\,W,$$

only 56 needs it.

We need not now go through the work while W′ travels from 6 to 12, because of the symmetry. We shall have the counterbracing of 67 settled when W′ is at 8, and it will be necessary if $W' > \frac{3}{2} W$; while for 89 it is settled when W′ is at 10, and is necessary only if $W' > 9\,W$.

In no circumstances can 12 and 10.11 require counterbraces (though practically they would be fitted in most cases), while if 56 and 67 do not require counterbracing, much less will 34 and 89.

We can now then give a general method of proceeding for any number of bays based on what we have just seen, as follows : Commence at the centre bays, because if they do not need counterbraces, the outer ones will not do so. To decide whether the first bay from the centre needs the extra brace, consider the stress when the travelling load is at the outside extremity of that bay (from the centre outward), because it then produces its greatest compressive effect on that diagonal. If the first bay require counterbracing, go on to the second, taking the load at its extremity ; and so on till a bay is found

which needs no counterbrace; and we can then be sure that none of the outer bays will need it either.

The foregoing of course only applies to the case in which all the W's are equal, and the original bracing as in Fig. 248; for other cases the work should be gone through as we did in the first instance.

The effect of the passage of a long distributed load, as a train, is rather more complex, and we must refer for it to the larger treatise.

EXAMPLES.

1. A lattice girder 20 feet span, 3 feet deep, in four divisions, is loaded at the top of the left hand quarter span with 5 tons, and supported at the ends of the bottom boom. The braces are stout flat bars. Find the stresses in the bars.

Ans. Counting from the left—uprights, $-1\frac{7}{8}$, $-2\frac{1}{2}$, 0, 0, $-\frac{5}{8}$ tons; diagonals, ± 3.64, ± 1.21, ± 1.21, ± 1.21; top boom, 3.12, 5.21, 3.12, 1.04 compression; bottom boom, same as top, but tension.

2. How would the preceding results be modified if the braces were slighter, so that the compression brace only took half as much load as the tension brace?

Ans. Uprights, $-2\frac{1}{2}$, $-3\frac{1}{3}$, 0, 0, $-\frac{5}{6}$; diagonals, $+4.84$, $+1.61$, $+1.61$, $+1.61$, -2.42, $-.81$, $-.81$, $-.81$; top boom, $-4\frac{1}{6}$, $-5\frac{5}{9}$, $-3\frac{17}{36}$, $-1\frac{7}{18}$; bottom boom, $+2\frac{1}{12}$, $+4\frac{31}{36}$, $+2\frac{7}{9}$, $+\frac{25}{36}$.

3. Find the stresses produced when a bridge platform, which rests on the bottom booms of two girders, similar to that of (1), is loaded with $\frac{1}{2}$ ton per foot run.

Ans. Taking the bars in the same order as (1), $-\frac{15}{16}$, $+\frac{5}{8}$, 0, $+\frac{5}{8}$, $-\frac{15}{16}$; ± 1.82, ± 1.21, ± 1.21, ± 1.82; 1.56, 3.66, 3.66, 1.56 compression, and the same tension for bottom boom.

4. Find the greatest stress in each bar of the bridge of question (3) when a travelling load of 10 tons crosses it.

Ans. $-2\frac{13}{16}$, $+3\frac{1}{8}$, $\pm\frac{5}{8}$, $+3\frac{1}{8}$, $-2\frac{13}{16}$; ± 5.46, ± 6.1, ± 6.1, ± 5.46; ∓ 5.2, ∓ 11.5, ∓ 11.5, ∓ 5.2, for top and bottom booms respectively as before.

5. What would be the effect on the results of the preceding questions, if the trusses were made 4 feet deep instead of 3 ft.?

Ans. The stresses in the bars of the boom would be $\frac{3}{4}$ of those found before. In the diagonals the stresses would be reduced in the ratio of 59 : 64 nearly.

6. An N girder in 10 divisions is loaded with 1 ton at each joint. A load of 16 tons travels along it. Find which bays will require a counterbrace. *Ans.* The middle six.

7. In the preceding find the stresses in all the bars, assuming the counterbraces equally efficient against tension and compression when the load is at the centre.

CHAPTER XVIII

BENDING OF SOLID BEAMS

WE have seen that in a girder the bending is resisted by the booms, and the shearing by the diagonals, and this justifies our original separation of the effect of a load into these two actions, since they are resisted by distinct pieces, and H is independent of F while S is independent of M.

We can from the preceding obtain a preliminary idea of how one kind at least of solid beam resists bending,

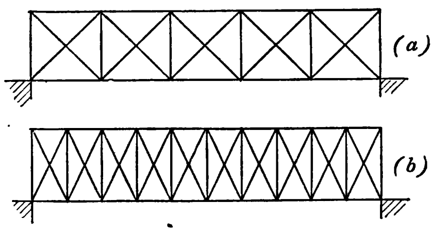


Fig. 250.

for—the above conclusions are true, no matter how long or short the bays may be; thus they are equally true for girders either as (*a*) or (*b*), Fig. 250. And they would still hold even if we reduced the lengths of the bays so much that the uprights and diagonals actually came into contact with each other. This being so, we have only now to change simple contact into connection to obtain

a continuous solid beam consisting of two booms or flanges, united by a solid sheet of metal called the web, as Fig. 251.

If then we assume that the connection of the bars into a solid does not materially alter the distribution of the forces, we get as a first idea of the resistance of such a solid beam to transverse loads, that the flanges resist the bending, the stress in either being H, given by

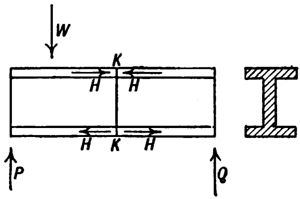


Fig. 251.

$$Hh = M,$$

where M now will be the moment at the section at which we wish to find the value of H. Since the boom is now supposed to be made up of an indefinitely large number of short rods, for each of which the value of H is different, there will be a continuous variation of H from point to point, and the opposite joint is now in the same plane as its indefinitely short bar, hence M is taken at the section as above stated.

In a girder as Fig. 251 the top flange would be compressed and the bottom extended; and at any given section KK we should have the forces H, H, and H, H as drawn. These forces we represent by arrows drawn in the opposite directions to those we used in the framework girders, because there we put the arrows on the bar and they represented the action of the bar on the rest of the structure; here the bar is indefinitely short, and we cannot put arrows on it, so we put them as shown; hence they represent actions exerted on the bar at K, or since this is simply a point, they represent the mutual actions of the two parts of the flange on each other (compare chap. xvi. page 313). We then have

$$Hh = M_K,$$

H being the total stress at K in either flange, so that if

$$A_t = \text{sectional area of top flange,}$$
$$A_b = \quad ,, \quad ,, \quad \text{bottom flange.}$$

Intensity of stress is, on top flange, H/A_t; on bottom flange, H/A_b. Then, by our assumption, the shear F_K is distributed over the sectional area of the web only, and hence, assuming it to be uniformly distributed,

$$\text{Intensity of shear in web} = \frac{F_K}{A_w},$$

where $\qquad A_w = \text{sectional area of web.}$

The assumptions here made give results for a beam of I section, useful for rough calculation; but we can see at once that they will not be even approximate for all beams. For example, some beams have no flanges at all, yet we know they can resist bending, while the preceding formula would give them no resistance to bending. We must then proceed to a closer investigation for solid beams, and as usual we will consider one action only at a time. We will commence then with bending, and investigate the stresses produced in a solid beam when subjected to bending only, or

Pure Bending.—If a beam be subject to pure bending, the shear at every point must be zero, and we must first then see what kind of loading is necessary to produce this result.

Take now a beam CD (Fig. 252), loaded at C and D with equal loads W, and supported symmetrically at A and B. Then the supporting forces at A and B will be each W.

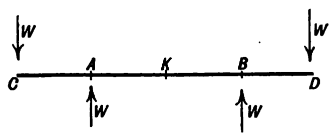


Fig. 252.

Taking now any point K between A and B we have

$$F_K = W - W = 0,$$

so the part AB is everywhere free from shear. Also

$$M_K = W \cdot CK - W \cdot AK,$$
$$= W \cdot AC.$$

So that not only is there pure bending but also constant bending all along the part AB.

But to produce the above effect it was not necessary to consider equal loads at C and D, for let them be W_1 and W_2 (Fig. 253).

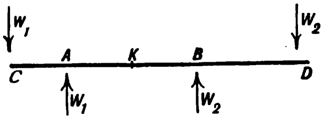


Fig. 253.

Now find A and B such that the supporting forces shall be W_1 at A, W_2 at B. To satisfy this condition we must have, taking moments about C,

$$W_1 AC + W_2 \cdot CB = W_2 CD,$$
$$\therefore W_1 \cdot AC = W_2 \cdot BD.$$

And this is the only condition necessary. We shall then have for all points in AB as K—

$$F_K = 0$$

and

$$M_K = W_1 AC, \text{ or } W_2 \cdot BD,$$

so there is pure and constant bending all along the part AB.

We can now see that in order to have $F = 0$ over any part of a beam, it is necessary that the forces outside each end of the part should reduce to two equal and opposite couples, one at each end; as $W_1 \cdot AC$ and $W_2 \cdot BD$; and we shall then have as a necessary condition that M will be constant along the part.

We have so far called AB a part of a beam, but we can if we please speak of AB as a beam, and consider the pieces CA, BD as outside bodies acting on it. If now we consider sections taken at A and B, then CA is in equilibrium under the couple $W_1 \cdot AC$, and the action on its end of the end of AB. Hence the action

of AB on the end of CA must reduce to a couple equal to $W_1 . AC$, and therefore the action of AC on the end of AB must reduce to a couple $W_1 . AC$, since the action between the ends of CA and AB is mutual (see page 262). Similarly the action of BD on the end B of AB is a couple $W_2 . BD$, and hence the beam AB is in equilibrium under the action of equal and opposite couples applied to its ends, which were present as in Fig. 254.

And we say finally that—

To produce pure or simple bending, a beam must be acted on by equal and opposite moments applied to its ends.

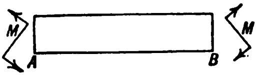


Fig. 254.

Bending in a Circular Arc—Neutral Axis.—When a beam is under pure bending, M is everywhere the same, and since it is M which produces the bending, evidently any one point will be bent just as much as any other. But the only curve which is equally curved at all points is an arc of a circle; hence we conclude that the originally straight beam bends into a circular arc.

Next, What stresses will this bending produce? To answer this question it is not sufficient to consider what happens to the beam as a whole, but we must examine the changes which take place in the small parts of which it is made up. Also we can only find an answer in the first instance for one particular kind of cross-section, or

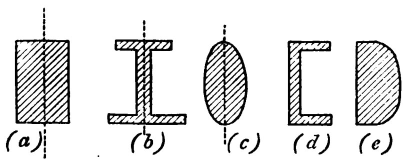


Fig. 255.

at least for cross-sections possessing a certain property, viz. symmetry about an axis in the plane of bending,

thus for beams bent in a vertical plane the sections must be such as (*a*) (*b*) or (*c*), but not (*d*) or (*e*) (Fig. 255).

Again, we cannot by considering the forces applied to the beam find the stresses on any section or the changes of shape of any small portion of the beam, simply by the use of the laws of Statics and Hooke's Law, since

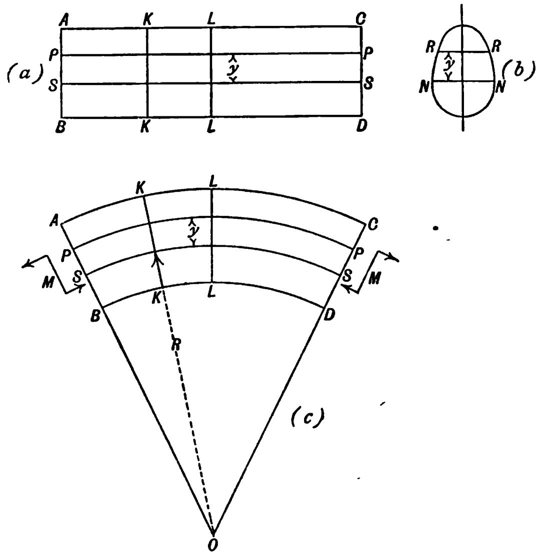


Fig. 256.

even if we could obtain equations they would be unsolvable. We are then obliged to proceed in the reverse way as follows :—

We assume a certain change of shape to occur—what kind to assume practical observation tells us—then we find the stresses which must accompany such change of shape, and finally we examine if these stresses and the external forces balance as they should do to satisfy

the laws of Statics. If the result is satisfactory according to the last test, then we accept it as the true result; for in all these problems we know that there can be only one result, *i.e.* a certain beam loaded in a certain way will always bend into exactly the same shape; hence a result which satisfies the above condition must be *the* result.

ABCD (Fig. 256) shows a side view and (*b*) a cross-section, this being taken to be uniform all along the beam.

Now imagine the beam divided up before bending into thin longitudinal layers as in (*a*), PP being one and RR in (*b*) showing the cross-section of PP. Then, when the beam is bent from (*a*) to (*c*) by the application of the couples M to its ends, all the layers will bend also.

Now first we can see at once that they all have a common centre after bending, viz. O, for otherwise instead of fitting one another as they actually do they would lie as Fig. 257.

Next we *assume*, that a plane cross-section such as KK in (*a*) remains also plane after the bending in (*c*). We could in a way prove this as follows:—

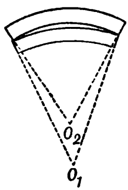


Fig. 257.

Taking LL the central section this undoubtedly remains plane, since there can be no reason tending to bend it one way more than the other. But the B. M. being constant all along, the stress will be distributed in the same way on all parallel sections (compare page 264), and hence, since this distribution of stress on LL leaves LL plane, it will also leave the other cross-sections plane.

This proof, however, contains assumptions, and hence it is better on the whole to make the original assumption at once, and suppose that, as in (*c*) AB, KK, LL, CD, all remain plane, and consequently all pass through O.

We require now one more datum before we can pro-

ceed to determine the distribution of the stress, which we can obtain either by theoretical reason or actual experiment. First, by reasoning—

There is no resultant longitudinal force acting on the beam, and hence as a whole it can neither be lengthened nor shortened. The layer AA is longer than BB in (*c*), and all the layers differ in length, but their mean length must be unaltered; or there will be some layer towards the centre of the beam as SS, whose length will be unaltered, this being the layer whose length is the mean of all.

By actual experiment also it is found that during bending AA extends while BB shortens, and hence, as before, some layer SS neither shortens nor extends.

This layer is of the greatest importance in the theory of bending, and it is called the **Neutral Surface** (its thickness being very small, it is of course only a surface). The Neutral Surface is shown in Fig. 256 by its two sections SS and NN; NN being its transverse section by any plane KK, and NN is called the **Neutral Axis** of the section.

[Note carefully, the beam has a neutral surface but not a neutral axis; often SS is taken as the neutral axis, being confused with the ordinary meaning of axis as a longitudinal line, but it is the section which has a neutral axis NN and not the beam at all.]

Distribution of Stress.—Let now R = radius of neutral surface. Take a slice PP, cross-section RR, distant y from the neutral axis.

Then PP was before bending (*a*) the same length as SS, but it is now longer than SS, and hence there must be a tension in PP.

To find its amount we have

$$\text{Extension of PP} = \text{PP} - \text{SS},$$

$$\therefore \text{ Strain of PP} = \frac{\text{PP} - \text{SS}}{\text{SS}}.$$

But

$$\frac{PP}{SS}=\frac{R+y}{R} \quad \text{(Fig. 256 (}c\text{))},$$

$$\therefore \text{ the strain } e=\frac{\frac{R+y}{R}-1}{1},$$

$$=\frac{y}{R}.$$

If now PP were a simple thin plate not connected to the other layers, the stress produced would be given us by Hooke's Law, and we assume that this is the case, *i.e. the stretching of each layer is assumed to be independent of its connection with the other layers.*

Let then p be the tension in PP, then

$$p=Ee=E\,.\,\frac{y}{R}.$$

Had we taken PP below the neutral surface we should have had compression produced, and if we take E to have the same value in compression as in tension, which is practically true, then the one formula

$$p=E\,.\,\frac{y}{R}$$

gives the stress at any point, y being taken negative for layers below the neutral axis, and the accompanying negative value for p indicating compression. We have now then obtained our first important result, regarding which we express in words thus :—

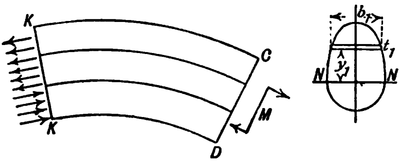
The stress at any point of the transverse section of a bent beam varies as the distance of the point from the neutral axis of the section.

Here for stress along the layer we put the correct expression, *i.e.* stress at any point of a transverse section of the layer.

The distribution of stress we have found applies to all transverse sections, and therefore to the ends; and consequently it follows that for our work to hold

true everywhere, the moment M should be applied to the ends of the beam, by forces applied all over the ends varying as their distances from the neutral axis. Actually of course the moments are not so applied, being originally derived from pressures of weights and supports, hence very near the ends our equations do not hold; but the lateral connection of the layers causes the distribution of stress rapidly to adjust itself to the kind we have found (compare page 264).

Position of Neutral Axis.—We have found an equation for p at any point, but it depends on the position of the point relative to the neutral axis, and at present we do not know where the neutral axis of any



(*a*) Fig. 258. (*b*)

given section is—all we know about it is one of its properties: we proceed then to determine its position.

Referring to Fig. 256 (*c*), the section KK divides the beam into two parts; we will now consider the equilibrium of one of these, say KCDK (Fig. 258).

KCDK is acted on by M at CD, and the stresses over the section at KK.

Resolve now at right angles to KK. Then

$$\text{Resultant stress on KK} = \text{resultant force on CD,}$$
$$= 0,$$

since a couple has no resultant.

Take now a small slice of the section, Fig. 258 (*b*), breadth b_1, thickness t_1, distance from NN y_1. Then

$$\text{Stress on slice} = p_1 b_1 t_1.$$

But

$$p_1 = \frac{E \cdot y_1}{R},$$

$$\therefore \text{ Stress on slice} = \frac{E}{R} y_1 b_1 t_1.$$

Dividing the whole area into such slices, we have

$$\text{Total stress on area} = 0,$$

$$\therefore \frac{E}{R} \cdot y_1 b_1 t_1 + \frac{E}{R} y_2 b_2 t_2 + \frac{E}{R} y_3 b_3 t_3 + \ . \ . \ . \ + = 0,$$

$$\therefore b_1 t_1 \cdot y_1 + b_2 t_2 \cdot y_2 + \ . \ . \ . \ + = 0,$$

i.e. the sum of the moments of all the strips about the neutral axis is 0, for

$$b_1 t_1 = \text{area of strip},$$

$$\therefore b_1 t_1 \times y_1 = \text{its moment about NN}.$$

But this can only be true when the axis of moments passes through the C. G. of the section ; also we took the slices, of which the neutral surface is one, at right angles to the plane of bending, and hence—*The neutral axis of a section passes through the C. G. of the section at right angles to the axis of the section, i.e. the axis of symmetry.*

C. G. is the usual abbreviation for centre of gravity.

We have now then the means of obtaining the stress at any point of the section of a beam bent into a circle of given radius. The chief portion of the work will in most cases consist in the determination of the position of the neutral axis, and we will examine in what manner this may best be done.

Neutral Axes of Various Sections.—If the

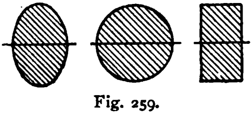
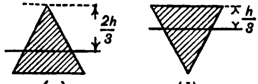


Fig. 259.



(a) (b)

Fig. 260.

section be rectangular, circular, or elliptic, the neutral axis passes through the centre of depth (Fig. 259).

If the section be triangular (Fig. 260), the axis is in (*a*) at $\frac{2}{3}$ of the depth, in (*b*) at $\frac{1}{3}$. For a trapezoidal section, as ABCD (Fig. 261), we must divide up into parts, of which the C. G.'s are known, *i.e.* parallelograms and triangles. In the present case divide into a parallelogram ACBE, and a triangle CED.

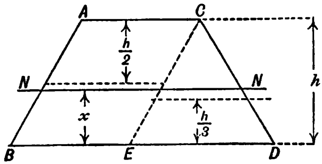


Fig. 261.

Then if NN be the neutral axis, and x its distance from AD, the area balances round NN, and we have

$$\text{ABCE}\left(\frac{h}{2}-x\right)=\text{CED}\left(x-\frac{h}{3}\right),$$

whence in any given case x may be determined.

For a T beam (Fig. 262) we divide into two rectangles—flange and web; the C. G. of each is at its centre of depth, and we take moments about NN as above.

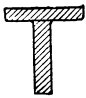


Fig. 262.

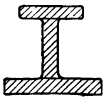


Fig. 263.

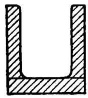


Fig. 264.

Similarly the I beam (Fig. 263), or channel iron (Fig. 264), would be divided into three rectangles.

In many cases beams are built up of plates and angle bars, sections of such being shown in Fig. 265. We must divide them up in the way which makes calculation easiest.

In some cases calculation is facilitated by taking moments about some other line than NN.

For example, take the case of the channel iron (Fig 266), and take moments about the base, then—

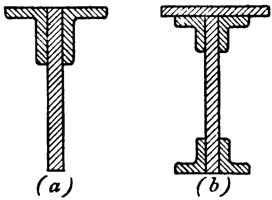


Fig. 265.

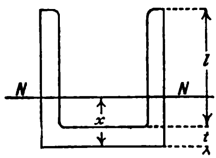


Fig. 266.

Moment of whole section = moments of parts.

Let

A = area of each web,
B = „ „ flange,
l = length of web,
t = thickness of flange.

Then

$$2A\left(\frac{l}{2}+t\right)+B\cdot\frac{t}{2}=(2A+B)x,$$

whence we obtain x directly.

Value of I Section.—When a bar is extended, the stress over any transverse section is uniform, and the whole of the fibres of which we may imagine the bar to consist help equally to resist the loading. We can thus have the limiting stress allowed, say f, fully reached at all points of a section.

But in bending the foregoing does not hold, for the stress varies from 0 at the neutral axis, each way, to say p_t tension at one edge, and p_c compression at the other, and thus the full powers of resistance of the inner fibres are not developed. It is then advantageous to so shape a section that as little material as possible may be left with its resistance undeveloped, *i.e.* we should remove the bulk of the material as far as possible from the neutral axis. This accounts for the adoption of the I and other sections in which the metal is principally concentrated in flanges. This we return to in the next chapter.

Beam of Equal Strength.—Continuing the above consideration, suppose first that the greatest stress allowed on the material either tension or compression is for definiteness, say, 4 tons per sq. inch. If the shape of the section be such that the neutral axis is at the centre of depth, then, when the beam is so much bent that there is 4 tons tension at one edge, there is 4 tons compression at the other, and hence the resistance of the metal on both sides of the neutral axis is developed as much as in bending it possibly can be.

But if the neutral axis were nearer the tension side, say, then, when the compressive stress was 4 tons, the tensile would be less, since

$$p \propto y,$$

but the beam must not be bent any more, otherwise the compression will be more than 4 tons, which is not allowed. Hence the metal on the tension side of the neutral axis has not its full resistance developed; and the full power of the beam is not put forth. Similar reasoning applies if the neutral axis be nearer the compression side.

It follows that the neutral axis should in this case lie in the centre of depth, and our sections should be shaped accordingly.

If, however, the values of f for tension and compression are as usual unequal, the best position of the neutral axis will not be at the centre of depth, but the same reasoning will show us where it should be. Let

f_c = compressive stress allowed,
f_t = tensile " "
y_c = distance of compressed edge from neutral axis,
y_t = " extended " "

Then, to fully develop the resistance of the metal, the neutral axis should be so situated that, when the bending is such that the stress at the tension edge is f_t, that at the compression edge should be f_c, when each part will be resisting as much as it possibly can.

This gives us at once the required position, for

$$\frac{f_c}{f_t}=\frac{y_c}{y_t}.$$

If then

$$h=\text{total depth of beam},$$
$$=y_c+y_t,$$

we have

$$\frac{y_c}{h}=\frac{f_c}{f_c+f_t},$$

and

$$\frac{y_t}{h}=\frac{f_t}{f_c+f_t},$$

so the position of the neutral axis is determined.

When the condition just found is satisfied, the section

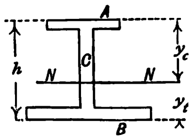


Fig. 267.

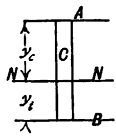


Fig. 268.

is said to be designed for equal strength. We will now apply this to the case of an I beam (Fig. 267). Let

A = area of top flange,
B = ,, bottom flange,
C = ,, web,
h = given depth,

and we will suppose the beam bent as usual, so as to be convex below, and in consequence the top flange is in compression.

In some cases the flanges are thin compared with h, so their thickness may be neglected, and the section treated as if the flanges were simple lines (Fig. 268), giving them, however, their proper sectional area in calculating.

Then, NN being the neutral axis for equal strength,

$$\frac{y_c}{y_t}=\frac{f_c}{f_t}.$$

But NN passes through the C. G., hence

$$\text{Total moment about NN} = 0.$$

Now

$$\text{Moment of A} = Ay_c,$$
$$\text{,,} \quad B = -By_t,$$
$$\text{,,} \quad C = C\left(\frac{h}{2} - y_t\right),$$

the moment of B is negative, since B is below NN.

$$\therefore \; Ay_c + C\left(\frac{h}{2} - y_t\right) = By_t,$$

or

$$A \cdot \frac{f_c}{f_c + f_t} \cdot h + C\left(\frac{h}{2} - \frac{f_t}{f_c + f_t} \cdot h\right) = B \cdot \frac{f_t}{f_c + f_t} \cdot h,$$

$$\therefore \; Af_c + C\frac{f_c - f_t}{2} = Bf_t,$$

which is the relation which must hold between A, B, and C. For the full determination of A, B, and C to resist a given moment we must wait till the next chapter, but we have investigated here the above relation in order to show clearly that it depends entirely on the relations between the stresses at different points of the section, and not on their absolute amounts under a given bending moment. This it certainly shows, for so far we have not seen how to find what these amounts are.

EXAMPLES.

1. An I beam 18 ft. span is 12 ins. deep over all. Each flange is 6 ins. by $\frac{3}{4}$ in., and the web is $\frac{1}{2}$ inch thick. Find the total stress in one flange under its own weight. Material cast-iron. *Ans.* 1730 lbs.

2. The beam in the preceding carries 1 ton at the centre. Find the greatest total stress in one flange. *Ans.* 11810 lbs.

3. In the preceding questions, find in each case the maximum intensity of the shearing stress on the web. *Ans.* $73\frac{1}{3}$, 287 lbs.

4. A beam is 12 feet long, and is loaded with 4 cwt. at one end and 5 cwt. at the other. Find the positions of two supports 8 feet apart so that there may be pure bending between them. *Ans.* 2 ft.; $2\frac{2}{3}$ ins. from 4 cwt.

5. Find the diameter of the least circle into which $\frac{1}{4}$ inch steel wire can be coiled, the stress being limited to 6 tons per sq. inch. *Ans.* 45 ft. 2 ins.

6. A steel ribbon, width 8 times its thickness, weighing the same per foot as the wire in the preceding, is bent into a circle $\frac{1}{4}$ the diameter of the smallest coil there found. Find the stress produced. *Ans.* 7.7 tons per sq. inch.

7. Find the position of the neutral axis of a T beam; flange 4 ins. by $\frac{3}{4}$ in., web $8\frac{1}{4}$ inches by $\frac{1}{2}$ inch. If the beam be bent into a circle 4000 ft. diameter, the flange being nearest the centre; find the greatest tensile and compressive stresses. Material wrought-iron.
Ans. 1.9 ins. from centre of web; $3\frac{1}{4}$ and $1\frac{5}{8}$ tons per sq. inch.

8. If the ratio of tensile to compressive stress allowed in the preceding be 5 to 3; find what should be the proper thickness of the web for equal strength. *Ans.* .583 ins.

9. Find the neutral axis of an I beam. Top flange 6 ins. by $\frac{3}{4}$ in., bottom flange 10 ins. by $1\frac{1}{4}$ in., web 14 ins. by $\frac{5}{8}$ in.
Ans. 5.83 ins. from bottom.

10. Find the position of the neutral axis of a channel iron. Flange 8 ins. by $\frac{5}{8}$ in., each web $7\frac{1}{2}$ ins. by $\frac{1}{2}$ inch.
Ans. $2\frac{3}{4}$ ins. from outer edge of flange.

11. A beam is built up of a $\frac{5}{8}$ plate $15\frac{1}{2}$ ins. deep, to the top of which are riveted a pair of angle irons $3\frac{1}{2}$ by $3\frac{1}{2}$ by $\frac{1}{2}$ inch, and to the bottom a pair 4 by 5 by $\frac{5}{8}$ inch, the 4-inch side being riveted to the plate. Find the position of the neutral axis.
Ans. 6.78 ins. from bottom.

12. Find the neutral axis of a box girder. Top plate 14 ins. by $\frac{3}{4}$ in., bottom plate $\frac{1}{2}$ inch thick; each side plate 16 by $\frac{1}{2}$ inch; angle irons 3 by 3 by $\frac{3}{8}$ in.
Ans. 8 ins. from top.

CHAPTER XIX

BENDING (*continued*)

In the last chapter we found a relation between the stress produced and the radius of bending. What we usually require, however, is the stress produced by a given moment, hence we now proceed to consider this. The figure (258) on page 356 is here reproduced (Fig. 269), and we proceed to inquire further into the equilibrium of the piece KCDK.

In chap. xviii. we resolved the forces, so we will now take moments about some axis, *i.e.* consider its equi-

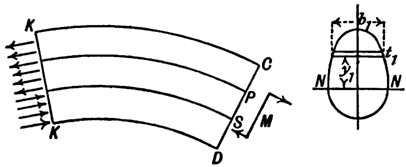


Fig. 269.

librium against rotation. The axis we will select is NN, its projection in the plane of (a) being the point R, so if we thought of the forces as all in the plane of (a) we should speak of taking moments about R; but since we know they are not all in the one plane we must not as in two dimensions speak of a point, but of an axis. We then have

$$\begin{matrix}\text{Moment of stresses on KK} \\ \text{about NN}\end{matrix} = \begin{matrix}\text{moment of couple M} \\ \text{about NN.}\end{matrix}$$

But it is proved in Statics that the moment of a couple

about all points in its plane—*i.e.* strictly about all axes perpendicular to the plane—is the same,

$\therefore$ Moment of stresses on KK about NN = M.

To find the moment of the stresses, divide the section up into small slices parallel to NN as before. Let

$$b_1 = \text{breadth}, \quad t_1 = \text{thickness},$$
$$y_1 = \text{distance from NN}$$

of one of these slices.

Then the intensity of stress on it is p_1, and

$$\therefore \text{Moment of stress on strip} = p_1 b_1 t_1 \times y_1,$$

p_1 is constant since t_1 is very small. But

$$p_1 = \frac{E}{R} y_1,$$

$$\therefore \text{Moment of stress on strip} = \frac{E}{R} y_1^2 b_1 t_1.$$

Hence, taking all the strips,

Total moment about NN

$$= \frac{E}{R} b_1 t_1 \cdot y_1^2 + \frac{E}{R} b_2 t_2 \cdot y_2^2 + \frac{E}{R} b_3 t_3 \cdot y_3^2 + \ldots .$$

[There is now no difference of sign between moments of stresses above and below, for they all turn clockwise; this is also shown by all the y's being squared, so it is immaterial whether they be plus or minus.]

We have then

$$\frac{E}{R}\{b_1 t_1 \cdot y_1^2 + b_2 t_2 \cdot y_2^2 + \ldots +\} = M.$$

The expression inside brackets is very similar to one we have met with before (page 213); it represents the sum of small areas multiplied by the squares of their distances from an axis, while that on page 213 is the same, substituting weights for areas.

We denote the expression then by the same name as before, viz.—Moment of Inertia, and denote it by the letter I. So

$$b_1t_1 \, . \, y_1^2 + b_2t_2 \, . \, y_2^2 + \ldots + = \text{Moment of inertia of area of section about NN,}$$
$$= \text{I}.$$

[Notice this has nothing to do with force, but is a purely geometrical quantity.]

Substituting then in the above equation, we obtain

$$\frac{\text{E}}{\text{R}} \, . \, \text{I} = \text{M},$$

or

$$\frac{\text{E}}{\text{R}} = \frac{\text{M}}{\text{I}},$$

but

$$\frac{\text{E}}{\text{R}} = \frac{p}{y}, \qquad \text{(chap. xviii.)}$$

and hence

$$\frac{p}{y} = \frac{\text{E}}{\text{R}} = \frac{\text{M}}{\text{I}},$$

equations giving us a complete relation between all the quantities involved in the bending of a beam as described in the last chapter.

We ask now, Will these equations hold for ordinary cases of bending? And first we must carefully remember that they have nothing to do with the breaking of a beam by bending; as stated on page 267, the whole of our work refers to metal in the elastic state, and when the stress passes the limit of elasticity our work at once breaks down. We must not then think that if f_u be the ultimate strength of a metal in tension, the moment which will break a beam will be given by

$$\text{M} = \text{I} \, . \frac{f_u}{y},$$

because there is then no such relation between M and f. We specially note this point, as it is one often seized upon by those who do not comprehend the true nature of the formulæ, in order to throw doubt on their correctness.

We pointed out on page 356 the effect of M not being

applied in the exact way necessary, and there remains only to consider the effect of M not being constant, or of the sectional area or depth not being constant.

These we can consider together, for suppose Fig. 270 represents a beam, in which all three quantities vary from point to point. Take now a section at K. Then, if we take a very small length of the beam at K, M will be M_K and will not vary, and neither will the area nor depth.

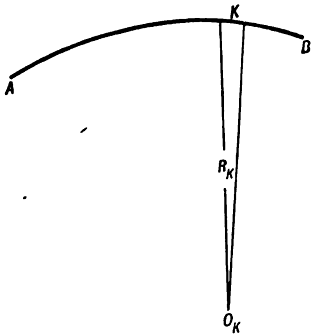


Fig. 270.

Now there was no restriction in our work on the length of the beam, and hence the equations hold for as short a beam as we please to consider. Therefore the indefinitely small length of beam at K bends into a circular arc of radius R_K, given by

$$\frac{E}{R_K}=\frac{M_K}{I_K},$$

and the stress at any point of the section at K is given by

$$\frac{p}{y}=\frac{M_K}{I_K}.$$

R is called the radius of curvature at K, and both it and p vary from point to point along the beam, but at any section

$$\frac{p}{y}=\frac{M}{I}=\frac{E}{R},$$

the various magnitudes being calculated at the section.

When M varies, we shall always find that there is shearing as well as bending, but the effect of this is considered separately; and then if the true effect be

required, the two separate effects must be compounded ; the method of doing this is outside our present scope. The equations above, then, are taken to be true in all cases of bending for beams of symmetrical sections (page 351), and their truth can be verified by measurement of the curvature of beams bent by given loads ; such experiments amply prove the practical truth of the laws, and hence justify us in the assumptions with which we were obliged to commence. (See note at end of ch. xxi.)

Moments of Inertia.—The calculation of I is an important part of the work to determine p, so we will consider now some points bearing on it. We have

$$I = b_1t_1y_1^2 + b_2t_2y_2^2 + \ldots,$$

an expression, as we have said, identical in form with that of page 213 ; hence the algebra of summing it will be identical with that of that page, and since there we put

$$w_1r_1^2 + w_2r_2^2 + \ldots = Wr^2,$$

here we can put

$$b_1t_1 \cdot y_1^2 + b_2t_2 \cdot y_2^2 + \ldots = Ar^2,$$

since b_1t_1, etc., are small areas, A representing the area of the section ; and r will be the radius of gyration of the section round the axis NN, since y_1, y_2, etc., would be the r_1, r_2, etc., of rotation round that axis.

Hence the values of r^2 may be taken from the table on page 214.

In this part of the work it is convenient to express r^2 in terms of h^2—h being the depth of section—and we put for r^2, nh^2, and then the fractions of page 214 give the values of n. Thus for a circular section $n = \frac{1}{16}$, etc.

There is a difference between the axis of an ordinary rotation, as that of a fly-wheel, and the axis about which our present moments are calculated. The axis in our present case is in the plane of the section, while in the case of a rotating fly-wheel the axis about which the

moment is taken is perpendicular to the plane of the wheel. The rotation which would give our present radius of gyration would be such as that of a circular throttle-valve, not of a wheel.

There are certain properties connecting moments of inertia of sections about various axes which are useful and can be obtained without the use of calculus, and we will consider one or two of them.

Polar Moment of Inertia.—First, consider a relation bearing on the point mentioned above.

Let the irregular figure (Fig. 271) be a plane area, a a small area forming part of it. Take O any point, and draw two axes OX, OY through it at right angles; also imagine a third axis through O at right angles to the plane of the paper.

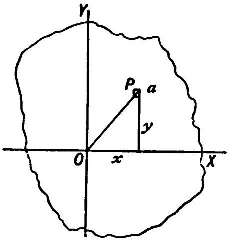


Fig. 271.

Join OP.

Then about the third axis,

$$\begin{aligned} \text{M of I of } a &= a\,.\,\text{OP}^2, \\ &= a(x^2+y^2) = ax^2+ay^2, \\ &= \text{M of I about OY} + \text{M of I about OX}. \end{aligned}$$

If then we take all the small areas making up the whole, we have, adding all the results,

$$\begin{matrix}\text{M of I of all the} \\ \text{small areas about O}\end{matrix} = \begin{matrix}\text{Sum of M of I's about OY} \\ +\text{sum of M of I's about OX},\end{matrix}$$

or

$$\text{I of whole area about O} = \text{I about OY} + \text{I about OX}.$$

Here I about O stands for I about an axis through O perpendicular to the plane.

This is generally called the Polar Moment of Inertia.

Now OX and OY were taken in any direction, the only condition being that they be at right angles, and hence we have—

The polar moment of inertia of an area about any point in its plane is equal to the sum of the moments of inertia about any two axes at right angles, through the point, and in the plane.

Apply this, for example, to a circular section radius r. We have

$$\text{Polar moment about centre} = A\frac{r^2}{2}.$$

From symmetry

$$\text{I about OX} = \text{I about OY},$$
$$\therefore \text{ I about any diameter} = \tfrac{1}{2} \text{ polar moment},$$
$$= \frac{Ar^2}{4}, \text{ or } \frac{Ah^2}{16}.$$

as given in the table.

Moments of Inertia about Parallel Axes.—When we speak simply of the moment of inertia of a section without mentioning any axis, the neutral axis is understood. This moment we denote by I_o.

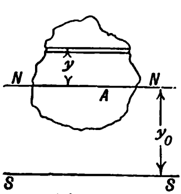


Fig. 272.

Let now A (Fig. 272) be an area of any shape, NN its neutral axis, SS an axis in the plane, parallel to NN and distant y_o from it. We will calculate the I about SS.

Divide A into strips parallel to NN as usual. Then, taking one strip, distant y from NN,

$$\begin{aligned}\text{I of strip about SS} &= \text{area of strip} \times (y_o + y)^2,\\ &= \text{area } (y^2 + 2yy_o + y_o^2),\\ &= \text{area} \times y^2 + 2y_o \times \text{area} \times y + \text{area} \times y_o^2,\\ &= \text{I of strip about NN} + 2y_o \times \text{moment of strip}\\ &\qquad \text{about NN} + \text{area of strip} \times y_o^2,\end{aligned}$$

therefore, taking all the strips and adding up,

$$\begin{aligned}\text{I of A about SS} &= \text{I of A about NN} + 2y_o \times \text{moment of A about}\\ &\qquad \text{NN} + A \times y_o^2,\\ &= I_o + Ay_o^2.\end{aligned}$$

The second term being zero, since NN is through the C. G. of A.

This expression is of great value, because knowing I_o we can at once deduce the I about any parallel axis. Or in some cases it is easier to find I about some axis parallel to NN than about NN itself, and the formula then enables us to deduce I_o.

For example, take a rectangle. Then

$$\text{I about one end} = \tfrac{1}{3}\,Ah^2,$$

$$\text{distance } y_o \text{ between end and neutral axis} = \frac{h}{2},$$

$$\therefore\ I_o = \tfrac{1}{3}\,Ah^2 - A\left(\frac{h}{2}\right)^2 = \tfrac{1}{12}\,Ah^2,$$

or we could have proceeded *vice versâ*.

Moment of Inertia of any Area.—The values of r^2 given on page 214, and the proposition just proved enable us to find the I of any practical section. Take for example the trapezoidal section of page 358.

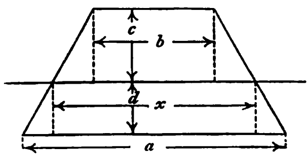


Fig. 273.

Find NN as explained on page 358, and then divide up as shown by the dotted lines. We have then to add together the I's of two rectangles about their ends, two triangles about their bases, and two triangles about axes through their vertices parallel to their bases. These are

$$\text{I of top rectangle} = \tfrac{1}{3}\,bc\,.\,c^2,$$
$$\text{I of bottom rectangle} = \tfrac{1}{3}\,xd\,.\,d^2,$$
$$\text{I of two top triangles} = 2\left\{\tfrac{1}{6}\left(\tfrac{1}{2}\,c\,.\,\frac{x-b}{2}\right)c^2\right\},$$
$$\text{I of two bottom triangles} = 2\left\{\tfrac{1}{2}\left(\tfrac{1}{2}\,d\,.\,\frac{a-x}{2}\right)d^2\right\},$$

. . . .

$$\therefore\ I_o = \tfrac{1}{3}\,bc^3 + \tfrac{1}{3}\,xd^3 + \tfrac{1}{12}\,(x-b)c^3 + \tfrac{1}{4}(a-x)d^3.$$

We must now find x from

$$\frac{x-b}{c}=\frac{a-x}{d},$$

the values of c and d are found on page 358, and we substitute. The section is not an important one, so we will not conclude the calculation; enough has been done to show the method.

Beams of I Section.—We saw in the last chapter that the I shape of section was advantageous, but this is still more shown by the work of the present chapter. For we now see that the resistance offered by the stress on any slice of a section depends on its moment about the neutral axis, and thus varies as the square of the distance of the slice from the axis, being $\frac{E}{R}bt\,.\,y^2$. We thus get a still weightier reason for removing the bulk of the material as far as possible from the axis, *i.e.* for the adoption of the I section.

For example, consider a rectangular section, and let us vary its shape to the I shown in dotted lines, the shaded rectangle of material being moved to the new position also shaded, and the three equal rectangles similarly treated.

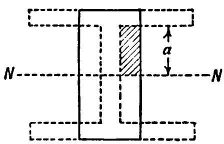


Fig. 274.

Then the shaded rectangle, area A, say, had in the rectangular section a moment of inertia $Aa^2/3$ about NN, while in the new position its moment of inertia is something more than Aa^2, and hence its power of resistance to bending is more than trebled. On the whole we have increased the M of I of the section by at least $4\left(\frac{2}{3}Aa^2\right)$; although its sectional area, and hence the amount of metal in the beam, is unaltered.

The process of removing metal into the flanges is of course limited by other considerations. In the first place we must have a web strong enough to withstand a

large portion if not all the shearing. And the web must not only be strong enough but also stiff, so that it may not give way under the compressive stresses which come on it, by buckling or sideway yielding; this effect is greater the greater the depth, and is in some cases, where the web is very deep, provided against by riveting angle bars at intervals to the web, as in Fig. 275, which shows what is known as a solid web bowstring girder, the vertical lines showing the stiffeners.

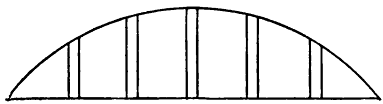


Fig. 275.

In Fig. 274 we show the increase of resistance effected while the depth remains fixed; but one way of increasing the distance of the metal from the neutral axis is by increasing the depth; in this case, however, we are limited by the necessity for stiffness, and thus the most economical depth is limited.

Again, the thinner the flanges the nearer is the metal in them to the outside edge, and hence the greater the moment of inertia, keeping their area constant. But again there is the necessity for stiffness in the compression flange to prevent buckling, and also, since the load is distributed over the width of the flange, it would, if too thin, bend as in Fig. 276.

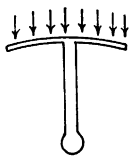


Fig. 276.

Further, there are practical questions to be considered, such as strength to withstand forces which come on the beam during manufacture; liability to blows, which necessitate *local* strength—*e.g.* a thin flange might be broken by a slight accidental blow, though quite strong enough when in place. Another consideration applying to cast beams, is that below a certain thickness it is not possible to secure a sound casting.

These questions we cannot here deal with, but we have

shown what considerations govern the question; and the student will thus, when he has attained the necessary *practical* knowledge, know how to deal with them.

Moments of Inertia of I Beams—Approximate Methods.—The accurate calculation of the M. of I. of an I beam is sometimes rather long, and a quicker method is often of practical value.

The most usual method of approximation is to treat the section as if all the metal of the flanges were concentrated on their centre lines.

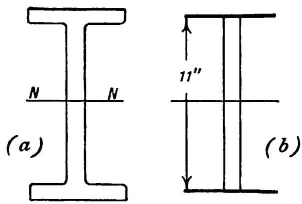


Fig. 277.

For example, take a section with equal flanges 6″ by 1″, and web 10″ by 1″ (Fig. 277, *a*); then we will treat it as if it were of the shape shown in (*b*), the flanges being represented as lines; still using in the calculation, however, their actual areas.

Then NN is the centre line, and from (*b*)

$$\begin{aligned} I &= \tfrac{1}{12}(11\times 1)11^2 + (6\times 1)\left(\frac{11}{2}\right)^2 + (6\times 1)\left(\frac{11}{2}\right)^2, \\ &= \frac{11^3}{12} + 3\times 11^2 = 111 + 363, \\ &= 474. \end{aligned}$$

We will now see what error is made, by finding the accurate value; and for this purpose we will use a method of subtraction, and not, as on page 371, of addition.

The I is the difference between a rectangle 12″ by 6″, and two rectangles each 10″ by $2\frac{1}{2}$″; and these all have the same neutral axis NN.

$$\begin{aligned} \therefore\ I &= \tfrac{1}{12}(12\times 6)12^2 - 2\times\tfrac{1}{12}(10\times 2\tfrac{1}{2})10^2, \\ &= 864 - 416\tfrac{2}{3} = 447\tfrac{1}{3}. \end{aligned}$$

There is then a considerable discrepancy between the

results, and we shall find that this is due to our having taken the web in (b) as if it extended between the centre lines of the flanges, *i.e.* length 11 inches. Let us now repeat the first calculation, but give the web its true dimensions, then

$$\begin{aligned}I&=\tfrac{1}{12}(10\times 1)10^2+363,\\&=83\tfrac{1}{3}+363=446\tfrac{1}{3},\end{aligned}$$

a result which is less than $\frac{1}{4}$ per cent in error; hence the second method should be followed in all cases if possible.

We will now consider how we should proceed to find the stress produced. We have

$$\frac{p}{y}=\frac{M}{I},$$

and for our section $y=h/2$, where h is the *total* depth, and not the depth of the approximate section b.

We can easily see that this is the proper value to take. For, accurately,

$$p=\frac{M\times 6}{447\tfrac{1}{3}} \qquad (1).$$

Using the approximate value of I, but taking 12″ as the depth,

$$p=\frac{M\times 6}{446\tfrac{1}{3}} \qquad (2).$$

While treating the question as if (b) were the real shape, and hence 11″ the total depth,

$$p=\frac{M\times 5\tfrac{1}{2}}{446\tfrac{1}{3}} \qquad (3).$$

Plainly (2) is practically accurate, while (3) is not so; and, moreover, it errs in the wrong direction, since it makes the stress appear less than its real value.

The latter inaccuracy would have been still worse had we found the value of I by the first method of approximation given; for this would give

$$p=\frac{M\times 5\tfrac{1}{2}}{474}=\frac{M}{86.2},$$

instead of

$$\frac{M \times 6}{447\frac{1}{3}} = \frac{M}{74 \cdot 5},$$

an error of 15.7 per cent on the unsafe side.

We will examine the present case also to see what amount of accuracy can be obtained by the use of the formula $Hh = M$.

Here, if we take h the mean depth from centre to centre of flange—*i.e.* 11 ins.—we obtain

$$H \times 11 = M,$$
$$\therefore \ H = \frac{M}{11}.$$

But H is the total stress on 6 sq. ins., *i.e.* $6p$

$$\therefore \ 6p = \frac{M}{11}, \quad \text{and } p = \frac{M}{66},$$

an error of 11.4 per cent, but on the safe side; the real p being less than this.

If we take, however, $h = 12$, then we get

$$p = \frac{M}{72},$$

which is practically accurate; the small error of about 3 per cent being again on the safe side.

The natural depth in this case would be 11 ins., that being the distance between the resultant stresses on the flanges, and then the 11.4 per cent error shows the amount of help afforded by the web. In the second case above, this is partially corrected by taking the outside value of h.

We have thoroughly examined the value of the approximations in this case, and from this the student will be able in any given case to judge of the best method. The approximations are of least use when the flanges are thick; and of course they should be used only for rough or preliminary calculations, final values of the stress being always obtained by the exact method.

We cannot always use the more correct approxima-

tion, because sometimes the areas are the quantities to be found, and we do not know the thickness of the flanges but only perhaps the total depth of the beam, or it may be the mean depth. In such a case we must of necessity treat the web as if its length were the full depth; but we now know what sort of an error that causes, and hence how to allow for it.

Resisting Powers of Flange and Web.—We can obtain a measure of the relative value of metal in the web and flange as follows:—

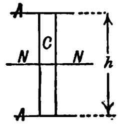


Fig. 278.

Fig. 278 represents an I beam, the approximate form being drawn. Let

A = area of each flange,
C = ,, web.

Then for a flange

$$I = A\frac{h^2}{4}.$$

For the web

$$I = C\frac{h^2}{12}.$$

Thus while A is multiplied by $h^2/4$, C is only multiplied by $h^2/12$, and hence area for area the metal of the flanges offers three times as much resistance as that of the web. By addition we obtain a simple approximate formula for I_0 of the beam,

$$I_0 = 2 \cdot A\frac{h^2}{4} + C \cdot \frac{h^2}{12},$$
$$= \frac{h^2}{2}\left(A + \frac{C}{6}\right).$$

If we consider the error of the approximation we see that we underrate A and overrate C, so that the ratio of powers is something in excess of three (compare with derivation of I section from rectangular, page 372).

Beams of Equal Strength.—We saw in the last chapter that for a section of equal strength or greatest

resistance the neutral axis (Fig. 268) must be so situated that

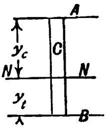


Fig. 279.

$$\frac{y_c}{y_t}=\frac{f_c}{f_t},$$

and hence

$$Af_c+C\frac{f_c-f_t}{2}=Bf_t.$$

We can now proceed to see how these areas A, B, and C can be calculated when we know the moment which the beam is required to resist.

We must first find I, and since we do not know the thicknesses, we must in Fig. 279 take the approximate figure. Then about the neutral axis.

$$\begin{aligned}
&\text{I of top flange} &&= Ay_c^2,\\
&\text{,, bottom flange} &&= By_t^2,\\
&\text{,, web} \quad \text{,,} &&= \tfrac{1}{3}\left(\frac{y_c}{h}\cdot C\right)y_c^2+\tfrac{1}{3}\left(\frac{y_t}{h}\cdot C\right)y_t^2,
\end{aligned}$$

here we split C into two rectangles, each having the neutral axis as one end, and their areas are $y_c/h\cdot C$ and $y_t/h\cdot C$.

$$\therefore\ I=Ay_c^2+By_t^2+\tfrac{1}{3}\frac{C}{h}(y_c^3+y_t^3).$$

Substituting the values of y_c and y_t from page 362,

$$\begin{aligned}
I&=Ah^2\left(\frac{f_c}{f_c+f_t}\right)^2+Bh^2\left(\frac{f_t}{f_c+f_t}\right)^2+\tfrac{1}{3}\frac{C}{h}\cdot\frac{f_c^3+f_t^3}{(f_c+f_t)^3}h^3,\\
&=\left(\frac{h}{f_c+f_t}\right)^2\left\{Af_c^2+Bf_t^2+\tfrac{1}{3}C\frac{f_c^3+f_t^3}{f_c+f_t}\right\}.
\end{aligned}$$

But

$$\frac{M}{I}=\frac{f_c}{y_c},\quad\text{or }\frac{f_t}{y_t},\quad\text{or }\frac{f_c+f_t}{h},$$

$$\therefore\ M=\frac{h}{f_c+f_t}\left\{Af_c^2+Bf_t^2+\tfrac{1}{3}C\frac{f_c^3+f_t^3}{f_c+f_t}\right\}.$$

The equation here found, combined with the equation of page 362, is not sufficient to determine the four quantities A, B, C and h; but, as we have seen, there are other conditions which limit the values of some of

these quantities, *e.g.* h is limited as we have seen on page 373; then, if we are given a certain value of h, this limits the thickness of the web so that C would be determined, and then the two equations would determine what should be the proportions of A and B; we should call this the best beam under the given conditions. In wrought-iron the T shape is common, since this iron is strong against tension; hence this gives us the extra condition $B=0$; and so for other conditions, examples of which will be given.

Beams of Uniform Strength.—If a beam be subjected to a constant B. M. its section everywhere should be the same; but in ordinary cases the B. M. varies from point to point, and the beams are accordingly made deeper or broader where the greater B. M.'s come. Plainly this course is economical, for take the case of a beam AB (Fig. 280) carrying a single weight W at C.

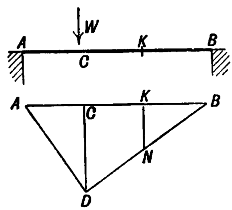


Fig. 280.

The diagram of B. M. is then a triangle ADB, where

$$CD = W\frac{AC \cdot CB}{AB} \qquad \text{(page 286).}$$

The section at C then must be strong enough to withstand this moment.

If then the section of the beam be uniform I and y will be constant all along AB; so that at K for instance the stress will be only KN/CD times what it is at C. But there is no gain by having only this small stress at K, because the beam will be injured if the stress at C pass the limit allowed, quite irrespective of what may be the stress at other points.

Moreover there is an actual loss by making the section at K larger than it need be for strength; for the

weight of the beam itself must in actual cases be considered, and hence there is a greater stress produced than there would be if the section at K were cut down to the least size necessary for strength.

For the best possible result we should so vary the section that the maximum stress f allowed is reached simultaneously at every point of the length. We have then no superfluous metal or weight anywhere, since the reduction of any section would cause the stress on that section to rise above f. When the section is so proportioned the beam is said to be of **Uniform Strength.**

We will now consider how the section should vary for a few simple cases.

Beam loaded at one End, fixed at the Other.—We can only deal with simple types of section as rectangular or circular. Let us take in the present case a rectangular section; then there are two cases, according whether we vary the depth keeping the breadth constant, or *vice versâ*. We will consider both these, taking first —Constant breadth. Let

$$b = \text{constant breadth.}$$

Take any section of the beam at KN (Fig. 281), and let

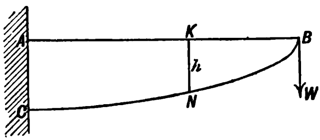


Fig. 281.

the depth at KN be h. Then at this section

$$\frac{p}{y_K} = \frac{M_K}{I_K}.$$

Then

$$y = \frac{h}{2}, \quad I_K = \tfrac{1}{12} bh^3, \quad M_K = W \cdot BK,$$

$$\therefore \frac{2p}{h} = \frac{12W.BK}{bh^3}.$$

Now p must be the same for all sections, whence, since

$$h^2=\frac{6W}{pb} \cdot BK,$$

we have

$$KN^2 \propto BK,$$

$\frac{6W}{pb}$ being constant.

But this curve is, we know, a parabola with B as apex and AB as axis; hence the profile CNB of the beam is a parabola.

To find its actual dimensions, let

$$W_m=\text{greatest load}, \quad f=\text{stress allowed.}$$

Then

$$AC^2=\frac{6W_m}{fb} \cdot AB,$$

which determines AC, and we then construct the curve

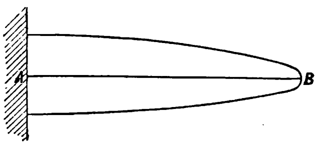


Fig. 282.

by the method given in the Preliminary Chapter.

The profile may be as in Fig. 281, or as here shown (Fig. 282), the depth at A being equal to AC in Fig. 281.

Consider now constant depth. Then, as before,

$$bh^2=\frac{6W}{p} \cdot BK,$$

which, since h is now constant, gives

$$b \propto BK.$$

and hence the plan is a simple triangle, as Fig. 283 (a).

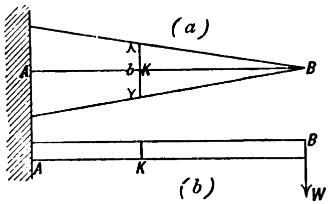


Fig. 283.

We will next consider the case of a—

Uniformly Loaded Beam fixed at one End.—This is a common practical case in balconies. Take a rectangular section (Fig. 284), and, as is always the case in practice, b constant. Then at K

$$M_K = \frac{w \cdot BK^2}{2},$$

$$\therefore \frac{p}{\frac{h}{2}} = \frac{w \cdot \frac{BK^2}{2}}{\frac{1}{12} bh^3},$$

whence, since all are constant but h and BK,

$$h^2 = \text{constant} \times BK^2,$$

$$\therefore \frac{h}{BK} \text{ is constant},$$

and the elevation is a triangle ACB.

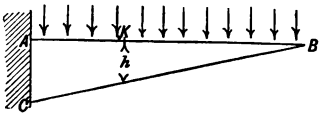


Fig. 284.

In actual practice the beam would not be brought to a point at B, but would be of the shape shown in Fig. 285, CD being straight.

Lastly, we will consider one case of a beam simply supported, viz.

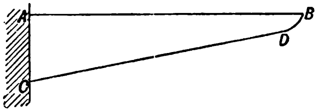


Fig. 285.

Beam loaded Uniformly, supported at Ends.—AB is supported at A and B (Fig. 286), and loaded with w lbs. per foot run. Let its section be rectangular of constant breadth b. We require to find its elevation.

The curve of BM is the parabola DFE of height $wl^2/8$.

Take a section at K, of depth h. Then

$$\frac{2p}{h}=\frac{12\mathrm{M_K}}{bh^3} \quad \text{(see preceding cases).}$$

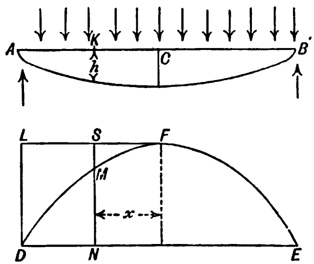


Fig. 286.

Let CK $=x$, C being the centre of AB, and AB $=l$. Then

$$\mathrm{M_K}=\mathrm{MN}=\mathrm{SN}-\mathrm{SM},$$

and

$$\frac{\mathrm{SM}}{\mathrm{DL}}=\frac{\mathrm{SF}^2}{\mathrm{LF}^2}=\frac{x^2}{\left(\frac{l}{2}\right)^2},$$

$$\therefore\ \mathrm{SM}=\frac{wl^2}{8}\times\frac{4x^2}{l^2}=\frac{wx^2}{2},$$

$$\therefore\ \mathrm{M_K}=\frac{wl^2}{8}-\frac{wx^2}{2}.$$

Substituting in the above equation for p, and making p constant, we obtain

$$h^2 = \text{constant} \times \left(\frac{l^2}{4} - x^2\right).$$

This is the equation to an ellipse with centre C and major axis AB. Hence the elevation is either as Fig. 286, ADB being an semi-ellipse; or equally either of

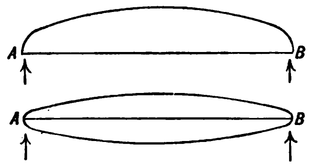


Fig. 287.

the forms of Fig. 287.

The depth at the centre is found by putting $p = f$ and $M = wl^2/8$, giving

$$\frac{2f}{h} = \frac{12 \cdot \frac{wl^2}{8}}{bh^3},$$

or

$$h^2 = \frac{3}{4}\,\frac{wl^2}{fb}.$$

EXAMPLES.

1. Obtain accurate results for questions 1 and 2 of the last chapter, thus showing what amount of accuracy is obtained by the approximate method there used.

Ans. 1680 lbs.; 11,400 lbs.

2. In question 5, page 291, find the necessary diameter of the axles that the stress may not exceed 3 tons per sq. inch.

Ans. $4\frac{1}{8}$ ins.

3. In question 7, page 309, find the stress produced at the centre of the crank pin by the bending action.

Ans. 5670 lbs. per sq. inch.

4. An I beam is 14 ins. deep, areas of bottom flange and web equal, each being four times the top flange, the area of which is 3 sq. ins. It is 20 ft. long, and supported at the ends. Find the greatest tensile and compressive stresses produced by a load of 6 tons placed 8 ft. from one end.

Ans. 2.06; 4.12 tons per sq. inch.

5. An I beam has flanges 4 ins. by 1 inch, and web 9 ins. by $\frac{1}{2}$ inch; span 10 ft. Find the greatest central load if the stress is limited to 5 tons per sq. inch. If the material be steel, how would the weight of the beam affect the result?

Ans. 7 tons; the carrying power would be less by 208 lbs.

6. Find the limiting span of a cast-iron pipe $9\frac{1}{4}$ ins. internal diameter, $\frac{3}{4}$ inch thick, the weight being 100 lbs. per foot length, and the stress not to exceed 2 tons per sq. inch.

Ans. 41 ft.

7. Compare the resistances to bending of a wrought-iron I beam, flanges 6 inches by 1 inch, web 8 ins. by $\frac{3}{4}$ inch, when upright and when laid on its side thus |——|.

Ans. 4.6 : 1.

8. Determine the weight which may be carried at the middle of a wooden spar 6 inches diameter, 15 feet long, supported at the ends; allowing a stress of 1500 lbs. per sq. inch.

Ans. 707 lbs.

9. In question 4, page 291, find the necessary width of tooth per H. P. Thickness at root $1\frac{1}{8}$ inch. Stress allowed 1000 lbs. per sq. inch. *Ans.* .2 ins. nearly.

10. In questions 1 and 2, page 308, find the value of the greatest stress produced. *Ans.* 78.5; 850 lbs. per sq. inch.

11. In question 3, page 308, the section of the beams is I shaped, equal flanges each four times as wide as it is thick, thickness of web one half that of a flange, and its area equal that of one flange. Find the necessary dimensions, stress not to exceed 3 tons per square inch. Neglecting the resistance of the web to bending, what shape should the elevation of the beam take, its width being uniform?

Ans. Thickness of flange $\frac{7}{16}$ in., and the others as given. The depth should vary as the B. M., hence the outline should be a parabola the same as the curve of B. M.

12. In questions 7 and 8, page 363, compare the resistance to bending of the two sections, per sq. inch of sectional area.

Ans. 1 : 1.24.

13. Find the moments of inertia of the sections given in questions 9, 10, 11 and 12 of the preceding chapter.

Ans. 968; 82; 1190; 1950 inch units.

14. In the preceding, find the weight each could carry at the centre of a 20 ft. span. Material in (9) cast-iron, tensile stress allowed 2½ tons, compressive 8 tons. In 10, 11, and 12, wrought-iron, tensile 5 tons, compressive 3½ tons.

Ans. 6.9; .89; 8; 14.2 tons.

CHAPTER XX

SHEARING AND TORSION

IN dealing with bending, the compound nature of the action is so evident that it is not necessary or useful to assign a *bending strength* to a material, but the strength of a beam is deduced from the values of the tensile and compressive strengths. (See note at end of ch. xxi.)

In the case of shearing, however, it is not so apparent that there is a dual action, and hence it has been, and still is practically, treated as a single action of a nature different from either tension or compression, and a metal is said to have a *shearing strength* just as it has a compressive or tensile strength.

We have seen in chap. xiv. (Fig. 185) the nature of the action called shearing, and how to calculate the shearing force on any transverse section of a beam. In the figure there given we have both bending and shearing, and it will be instructive to inquire what kind of forces are necessary to produce pure shearing, just as we have in the preceding chapter seen what produces pure bending.

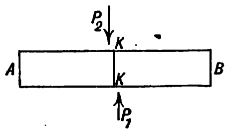


Fig. 288.

In order to have no bending moment at a given section a force must be applied in the line of the section, or indefinitely near to it on one side, as P_1 (Fig. 288). Then such a force can be balanced by an equal and opposite force P_2 acting

indefinitely near to the section on the other side. In the figure we have drawn P_1 and P_2 palpably out of line to show on which side of the section KK each is supposed to act, but each is supposed indefinitely near to KK, whence they are also indefinitely near each other, *i.e.* are in the same line and hence balance. The beam is now said to be in pure shear, and the shearing force is P_1 or P_2.

Then, considering the equilibrium of the right-hand piece, it is in equilibrium as shown under P_1 (Fig. 289), and the total shearing stress on the section KK, shown by the arrows, and plainly differing from tension or compression, being a sort of frictional action between the two surfaces at KK resisting relative sliding, instead of a direct pull or push acting normally to the surfaces. Hence

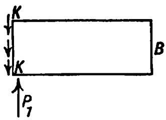


Fig. 289.

$$\text{Total shearing stress on KK} = P_1.$$

And assuming it to be uniformly distributed over the section, if

$$q = \text{intensity}, \quad A = \text{area of section},$$

$$qA = P_1, \quad \therefore \; q = \frac{P_1}{A}.$$

Pure shear is extremely rare in practice, *e.g.* in the shearing machine, bars after being cut can be plainly seen to have bent; the reason is that it is impossible to bring the forces P_1 P_2 indefinitely near each other; the jaws of the machine have a certain clearance, and in this distance the bar bends.

If the bar be both bent and sheared, we still have, referring to Fig. 289,

$$\text{Total stress on section} = F \text{ the shearing force},$$

but the value of q is altered, as we shall explain farther on.

Shear on Rivets.—The most usual example given of pure shear is that of a rivet connecting plates exposed

to a longitudinal pull. Fig. 290 shows a section and plan of a single riveted lap joint. Looking at the plan we see that the centre rivet shown must support the piece of plate between the dotted lines, which are half-way between that and the next rivet on each side. Each rivet has to support an equal strip, and hence we will consider the actions on one such strip and its rivets. Let

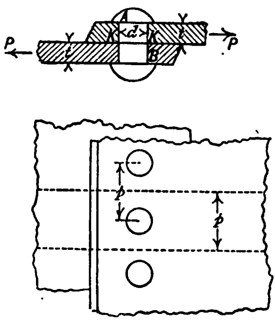


Fig. 290.

p = distance between the rivet centres or the **Pitch.**

Then p is the width of the piece of plate also. Let

P = pull on the strip,
t = thickness of metal,
d = diameter of rivet,
q = shearing stress on rivet,

these letters we will use right through the work.

Then, *if the rivet fit its hole tightly*, it will be under nearly pure shear at KK, and hence, its sectional area being $\pi d^2/4$,

$$q = \frac{P}{\frac{\pi}{4}d^2}$$

Why now must we insert the condition as to tight fitting? To answer this, consider how the forces P are applied to the rivet. They will be distributed over the rivet surfaces from A to K, and from B to K, so that the resultant forces P P will act roughly through the centres of AK and BK. But this being so, there will be a bending moment at KK, equal to P . AK/2 or P . BK/2, and consequently not pure shear. Now if the rivet bend it

must move in the holes, but if it fit tightly there will be great friction resisting such a motion; and this friction, if sufficient, will supply a friction moment equal and opposite to P . AK/2, and then at KK there will be pure shear. This we in all cases assume to be the case.

Pin Joints—Inequality of Shear.—In a pin joint (Fig. 291) the pin is subject to shear of total

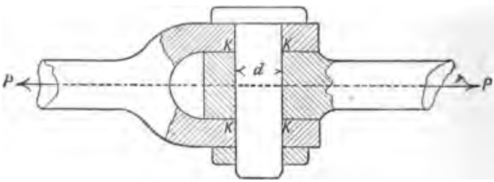


Fig. 291.

amount P over the two sections KK, KK; since, if it give way by shearing of the pin, both sections must be sheared, as shown in Fig. 292. But the pin will also be

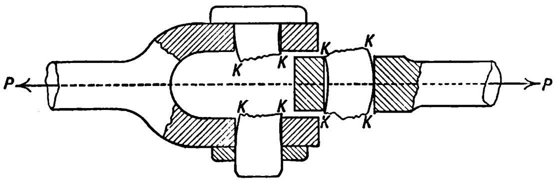


Fig. 292.

bent, as shown exaggerated in the figure, because the fit of the pin in the holes, and of the eye between the jaws, is a working fit, *i.e.* there is clearance in each of these. If the work be good the clearance will be small, but this will only lessen the amount of bending; in every case there will be some bending. Now the importance of the action lies not in the extra stress caused by the bending—which will as a rule be small—but in the fact that directly there is any bending at all, the

distribution of the shearing stress over the section is altered in a manner which can be theoretically determined. This determination we cannot enter into here, but the effect is that the stress at the centre of the pin, where it is greatest, is $\frac{4}{3}$ of the mean stress on the section, so that when the limit of elasticity is not exceeded,

$$\text{Shearing stress at centre of pin} = \frac{4}{3} q_{mean},$$

$$= \frac{4}{3} \frac{\frac{P}{2}}{\frac{\pi}{4} d^2} \quad \text{(there being two sections).}$$

If then f' be the greatest shearing stress allowed on the metal, we must put

$$f' = \frac{4}{3} \frac{\frac{P}{2}}{\frac{\pi}{4} d^2} = \frac{8P}{3\pi d^2},$$

and

$$d^2 = \frac{8P}{3\pi f'}$$

gives us the necessary diameter of pin to withstand the pull P.

Had the stress been uniform we should have found

$$d^2 = \frac{2P}{\pi f'}.$$

The effect here considered also holds in beams, the effect in a rectangular section being that

$$q_{maximum} = \frac{3}{2} q_{mean} = \frac{3}{2} \frac{F}{A}.$$

For other sections the ratio must be calculated by methods outside our present scope.

Riveted Joints. — For a full discussion of the proportions of riveted joints we must refer to works on the Design of Structures and Machines, to which subject

it properly belongs, but we may glance briefly at one or two questions concerning them.

Figs. 290 and 293 show examples of the two great classes into which these joints may be divided, viz. Lap (Fig. 290) and Butt (Fig. 293) Joints.

In the lap joint the two plates lap over each other, being connected by one row of rivets, as Fig. 290, or by two, three, or more rows (Fig. 297). To break such

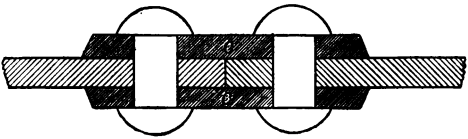


Fig. 293.

joints by shearing the rivets we have to shear each rivet across one section only, so that if there be n rows of rivets there will be n sections to be sheared for each strip of plate of width p. These rivets are said to be in single shear.

In the butt joint there may be double covering plates, or straps, as C, C, Fig. 293, or only a single one, as Fig.

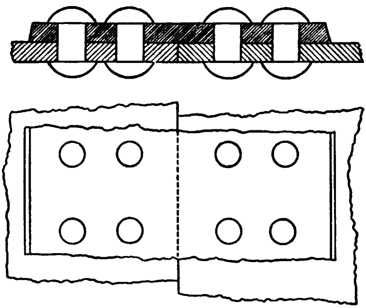


Fig. 294.

294. In the latter case we have simply a lap joint between the covering plate and each plate, so we treat it

as such. But with double straps there is the important difference, that each rivet must be sheared across two sections, or is in double shear (Fig. 295). If then there be n rows of rivets connecting the covering plate to

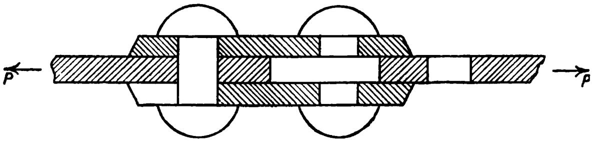


Fig. 295.

each plate, it is clear that $2n$ rivet sections must be sheared.

Let now P be the pull for one strip of plate, then we have, for lap or single strap butt joints,

One row, or single riveted $q=\dfrac{P}{\dfrac{\pi}{4}d^2}$.

Two rows, or double riveted $q=\dfrac{P}{2\dfrac{\pi}{4}d^2}$.

Three rows, or treble riveted $q=\dfrac{P}{3\cdot\dfrac{\pi}{4}d^2}$.

And so on, but there are rarely more than three rows.

For double strap butt joints, $q=$ one half the above values in each case.

In designing a joint we require to compare its resistance to shearing with that to direct tearing of the metal of the plate; and then, by making these equal, the joint will be just on the point of giving way to each at the same instant, and will be as strong as possible.

Consider then one strip of plate; this will, if torn, give way across the section through the rivet hole, that being its weakest section. This we could also see by consider-

ing the whole plate: it would plainly give way as in Fig. 296.

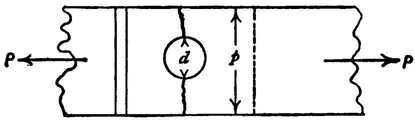


Fig. 296.

Thus we must consider the strength of this section, the area of which is $(p-d)t$. If then

$$f = \text{tensile stress allowed,}$$
$$f \times (p-d)t = \text{P}.$$

This is independent of the number of rows of rivets, for all that is necessary in order that a joint may break is that either plate give way along the row of rivets farthest from its edge (Figs. 296 and 297). It will not give way along any other row, because then, in addition to tearing the plate, one or more rows of rivets would be sheared. If now

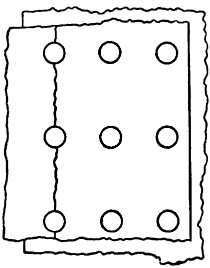


Fig. 297.

$$f' = \text{shearing stress allowed,}$$

and the number of rivet sections to be sheared be n, determined as we have seen, then

$$n \times f' \times \frac{\pi}{4}d^2 = \text{P}.$$

Whence, equating the two values of P,

$$f(p-d)t = f'n\frac{\pi}{4}d^2,$$

or

$$p = d + \frac{f'}{f}n\frac{\pi}{4}\frac{d^2}{t},$$

the equation which gives the pitch when d and t are known.

The value of t will be one of the data, and d is then determined almost entirely by constructive reasons, into which we do not enter. Also in some cases the pitch thus found cannot be used if the joint is to be water-tight, but this again is a purely practical consideration.

Efficiency of Joint.—By cutting a rivet hole through the plate, the strength of each strip is diminished from $pt \times f$ to $(p-d)t \times f$ or in the ratio $p - d \div p$. Hence the strength of the joint is $(p-d) \div p$ times that of the solid plate, and this ratio is called the Efficiency of the Joint (see page 273). Evidently it is advantageous to have p as large as possible.

Normal and Tangential Stress.—Our next step would be to consider the change of shape produced by shearing, and its connection with the stress. But it is convenient first to examine a little more closely into the nature of tangential or shearing stress.

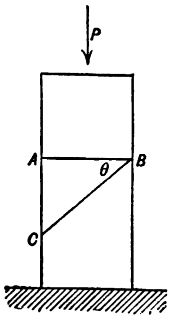


Fig. 298.

Fig. 298 represents a bar of rectangular section, thickness t at right angles to the paper, subject to a load P. If we take a section AB transverse, then there is compressive strength of intensity $P/(AB + t)$ over that section.

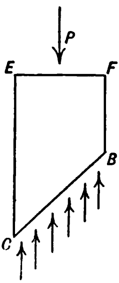


Fig. 299.

But now take an inclined section as BC (Fig. 298), making an angle θ with AB. Then, considering the equilibrium of the piece EFBC (Fig. 299), the stress on the section BC must be parallel to P, and its total amount equal to P; also it will be uniformly distributed (page 264), hence

$$\text{Intensity of stress on section BC} = \frac{P}{BC \times t},$$

and its direction makes an angle θ with BC, so it is neither normal nor tangential.

We can, however, resolve this stress into two components, one tangential or shear along BC, and the other normal or direct compression on BC.

For this purpose resolve the total stress P along and perpendicular to BC. The first gives

$$\text{Total tangential stress on BC} = \text{P} \sin\theta.$$

$$\therefore \text{ intensity, or } p_t = \frac{\text{P}\sin\theta}{\text{CB} \cdot t},$$

$$= \frac{\text{P}\sin\theta}{\text{AB}\sec\theta \cdot t},$$

$$= \frac{\text{P}}{\text{AB} \cdot t}(\sin\theta \cdot \cos\theta)$$

And the second gives

$$\text{Total normal stress} = \text{P}\cos\theta,$$

$$\therefore \text{ intensity } p_n = \frac{\text{P}\cos\theta}{\text{AB}\sec\theta \cdot t},$$

$$= \frac{\text{P}}{\text{AB} \cdot t}\cos^2\theta.$$

Now $\text{P}/\text{AB} \cdot t$ is the intensity of the stress on the section AB, *on which section it is purely normal;* hence, denoting this by p, we have

$$p_n = p\cos^2\theta, \quad p_t = p \cdot \sin\theta \cdot \cos\theta,$$

for the normal and tangential stresses on a section making an angle θ with the section AB on which the stress is normal.

The normal stresses are here compression; if P had been tensile the work would have been the same, but the signs of the normal stresses would have been altered, so they would be tensions; also the shearing stress would be in the opposite direction.

If we take a section at right angles to BC, the stresses p'_n, p'_t on it will be obtained by writing $\pi/2 - \theta$ for θ, hence

$$p'_n = p\sin^2\theta, \quad p'_t = p \cdot \cos\theta \cdot \sin\theta,$$

so that $p'_t = p_t$, or the tangential stress on planes at right angles is equal, a fact of great importance in the theory of strength of materials, to which we shall again recur.

Pure Shear.—In Fig. 300 ABCD is a block of material of thickness t, and we apply to it a compressive load of intensity p on AD, and therefore on BC to balance, and a tensile load of equal intensity p on AB and CD. Then on all sections parallel to AD there is pure normal stress of intensity p, and on all sections parallel to AB pure normal stress of intensity $-p$, the negative sign denoting tension, as opposed to compression, which we will consider positive.

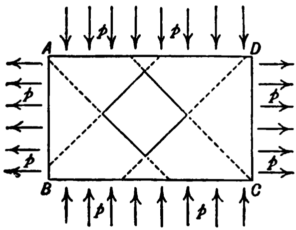


Fig. 300.

Draw now on the side of the block a small square with sides inclined at 45° to the sides AB, BC, and consider what action takes place on this small square of material, supposing it to extend through the whole thickness t.

Produce the sides as dotted.

Then, first, due to the compression p, there is, on each of the sections along the dotted lines, and thus on each side of the square,

$$\text{Tangential stress} = p \sin\frac{\pi}{4} \cdot \cos\frac{\pi}{4} = \frac{p}{2},$$

$$\text{Normal stress} = p \cos^2\frac{\pi}{4} = \frac{p}{2} \quad \text{(compression)},$$

as shown in Fig. 301.

Second, due to the tension p,

$$\text{Tangential stress} = \frac{p}{2},$$

$$\text{Normal stress} = \frac{p}{2} \quad \text{(tension)},$$

as Fig. 302.

The total effect of the tension and compression is now obtained by adding the separate results, whence we obtain the stress shown in Fig. 303, since the normal

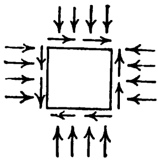


Fig. 301.

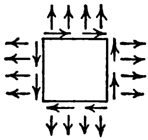


Fig. 302.

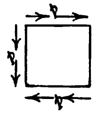


Fig. 303.

stresses cancel, while the tangential $p/2, p/2$ being in each case (Figs. 301 and 302) in the same direction, produce a tangential stress of intensity p along each side of the square.

Here then we have the square under pure shearing stress, and we see that this is of a dual nature, requiring for its production the existence of equal and opposite normal stresses in two directions at right angles, viz. directions parallel to AB and BC respectively; and hence in all cases of pure shear these two stresses p compressive and p tensile necessarily accompany it.

[We have started with the + and − stresses p, and proved they produce pure shear, because this shows a practical way in which it may be produced; but we can if we please start by assuming pure shear along the edges of such a small square, and then taking sections of the square parallel to AB and BC, it will be found that the stresses on them are pure tension and pure compression respectively.]

Equality of Shearing on Planes at Right Angles.—The preceding work proves the equality of shear on planes at right angles, which we have already noticed. For no other set of forces than those of Fig. 300 can produce pure tangential stress of intensity p on *any* side of the square, but this set produces equal tangential stress on *all* sides, which proves the result.

This being an important point we will consider it in another way.

Take ABCD, a small rectangle of material, thickness t, to which shear is applied along the edges AB and CD as shown (Fig. 304), the shear along CD being necessary to balance that along AB. Then if p_t be the intensity of the shear, there is a moment

$$p_t \times (\text{AB} \times t) \times \text{BC},$$

tending to turn the piece clockwise.

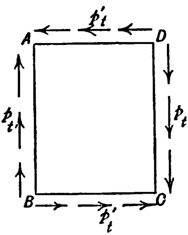


Fig. 304.

Now this cannot be balanced by the application of uniform normal stress to any of the faces, since the resultant stress on each face will pass through its centre. Hence to prevent the turning we must apply shear of intensity p'_t along the edges BC and AD. This will produce a turning moment

$$p'_t \times (\text{BC} \times t) \times \text{AB},$$

and since the two moments balance,

$$p_t \times (\text{AB} \times t) \times \text{BC} = p'_t \times (\text{BC} \times t) \times \text{AB},$$

or

$$p_t = p'_t.$$

Distortion due to Shearing.—We can now see what change of shape or strain accompanies shearing, for, referring to Fig. 300, the block will, under those stresses, elongate in the direction AD, and contract equally in the direction AB, hence taking the form of Fig. 305, in which the change of shape is shown in an exaggerated form.

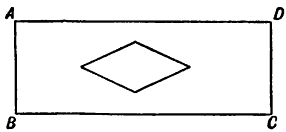


Fig. 305.

Hence the square of Fig. 300 now distorts into the rhombus of Fig. 305, the lengths of its sides remaining unaltered.

[The student must remember that all these changes of shape are extremely small; otherwise the last statement would not

hold, because if the side move sensibly, the stress is no longer pure shear along it, and changes of length will occur.]

Since the effect of pure shear is to produce distortion only, and no change of length in the direction of the shear—*i.e.* along the sides of the square—it is called a pure distorting stress, and its effect is measured by the angle of distortion, *i.e.* the angle through which the sides of the square move relatively to each other. If ϕ be this angle—*i.e.* the angles of the rhombus are $\pi/2 - \phi$, and $\pi/2 + \phi$—ϕ may be called the strain due to shearing, just as e is the strain due to tension. Also, just as we have the equation

$$p = \mathrm{E}e$$

proved by experiment to hold in tension, so we have the equation

$$p_t \text{ or } q = \mathrm{C}\phi$$

proved by experiment to hold in shearing.

This equation then connects together shearing stress and the strain produced by it, C being the modulus as E is, and being called the Modulus of Distortion, or of Torsion (why, we shall see directly), or the Coefficient of Rigidity, because it is a measure of the power of a material to keep its shape.

The value of C for wrought-iron and steel is about 10,500,000 for q in lbs. per sq. inch, and ϕ in circular measure.

[It is instructive to compare the preceding work with that of chap. xvii. on the shearing force in girders; the combination of tension and compression is identical in each case.

We also see now the necessity for stiffness in the web of an I beam irrespective of the bending stresses. For, taking sections at 45° to the shear, there is compressive stress in one direction, which will bend the web if it be not stiffened against it, just as the compressive brace (page 334) would bend if it were too thin.]

Torsion.—The chief value perhaps of the preceding work lies in its application to the principal practical case of pure shear, viz. torsion.

When a bar is considered as a whole torsion may properly be spoken of as a distinct kind of action, forming with tension, compression, bending, and shearing the five straining actions. Considered as an action on the particles of metal, however, only the first two are really distinct, and torsion is simply a case of shear.

[It hence finally reduces to tension and compression; but we do not follow it out to this, because we have treated shear as a distinct action, obtaining the equation

$$q = C\phi$$

for it experimentally, instead of attempting to deduce a relation from the extension and compression accompanying it.]

Let Fig. 306 represent a thin tube, fixed at AC, and having a twisting moment T applied to its end BD, the axis of T being that of the tube.

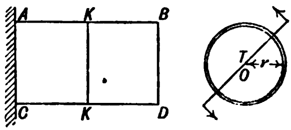


Fig. 306.

Then, taking any transverse section, at KK say, the section will from symmetry remain circular, and the effect of the moment would be, if the tube were cut at KK, simply to turn KBDK round its axis. In such a motion the surfaces at KK would simply slide over each other, and hence there is a simple tangential or shearing stress over the sections. We might equally consider a solid cylinder, we can easily see there is simply shear over a transverse section; but in the case of the thin tube we can now go on to consider the change of shape produced.

Torsion of a Thin Tube.—First, for the stress produced by T. Let

r = mean radius,
t = thickness (which is small),
q = intensity of stress over any section KK.

Then KBDK is in equilibrium under two sets of forces, as shown in Fig. 307 (*a*) on KK and (*b*) on BD.

[The moments appear to be in the same direction, when they could not balance, but this is because we are looking from opposite ends.]

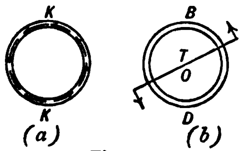


Fig. 307.

The total stress on KK is

$$q \times 2\pi r t,$$

and since each small stress is at right angles to the radius, the total moment is

$$q \times 2\pi r t \times r$$

(the arm r being common to all the stresses) about the axis of the tube.

This balances T on BD,

$$\therefore\ q \times 2\pi r^2 t = \mathrm{T},$$

and

$$q = \frac{\mathrm{T}}{2\pi r^2 t}$$

gives the stress on any transverse section.

Next, for the change of form, draw on the tube before

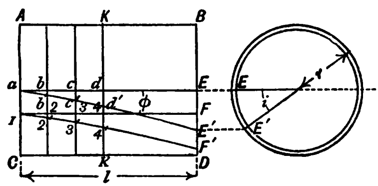


Fig. 308.

distortion a number of equidistant section lines as shown between AC and KK (Fig. 308).

[We take two for clearness, but there should be a great number.]

Also draw two lines $abcd$E, and 1234F longitudinally, and such that $a1 = ab$, thus forming a number of squares $ab12$, $bc23$, etc., which being very small may be considered as flat.

Then the small square $ab12$ is under shear q along $b2$ and $1a$, and hence it is under equal shear along ba and 12. The square then distorts into a rhombus $ab'2'1$, and the angle bab' or ϕ is given by

$$q = C\phi, \quad \text{or } \phi = \frac{q}{C}.$$

[The distortion being small b' still remains in the line $b2$, for ab' and ab differ only by an indefinite small quantity.]

The next square $bc32$ also distorts equally, but its side $b2$ is moved to $b'2'$ by the distortion of $ab12$, so that it takes up the new position $b'c'3'2'$; $b'c'$ making the same angle ϕ, as above, with bc.

Similarly $cd43$ moves to $c'd'4'3'$, $c'd'$ making an angle ϕ with cd.

We see then that the original straight line aE becomes distorted into a curve aE′, having the property that each small piece of it is inclined at the constant angle ϕ to its old position; the shape of aE′ is therefore that of a screw thread or helix. The same holds for 1F, which becomes the helix 1F′, and for all lines originally parallel to the axis.

If we look now at the end view, we see that E having twisted round to E′, the radius OE has twisted through the angle EOE′. This angle is called the **Angle of Torsion**, and is denoted by i.

We can now easily determine the relation between q and i. For let

$$l = \text{length of tube.}$$

Then, remembering that ϕ is a very small angle and EE′ a very small distance compared to l, we have

$$EE' = Ea \times \phi = l\phi \quad \text{(from Fig. 308, } a\text{)},$$

also

$$EE' = r \times i \quad \text{(from Fig. 308, } b\text{)},$$

$$\therefore\ ri = l\phi = l \times \frac{q}{C},$$

or

$$i=\frac{ql}{Cr}=\frac{Tl}{2C.\pi r^3 t}.$$

So we have found the stress and the twist produced by a given moment T.

Effect of Slitting a Tube.—Returning to the consideration of the small squares $ab12$, etc., we have seen they are under pure shear.

Hence it follows that in the directions of the diagonals there are tensile and compressive stresses q. There is tension *along* $a2$ which stretches to $a2'$, or tension *across* the section $1b$. Similarly *along* $1b$, or *across* the section $a2$, there is compression.

Suppose now we cut a slit along $a2$, then this would not interfere with the power of the square to resist the compression across this section, this simply forcing the

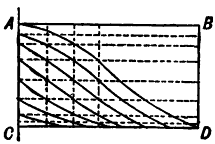


Fig. 309.

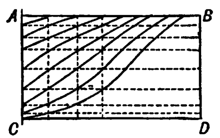


Fig. 310.

sides of the slit together. This also applies to all the small squares of which the tube is made up, since they are all identical as regards stress. If then we continue $a2$, cutting across each little square in succession, we get a helical slit at an angle of 45° all along the tube, and it appears the tube is not weakened as regards resisting T.

We could equally well cut another slit, and so on till we divide the tube up into a number of helical ribbons (Fig. 309), and yet its strength its unimpaired.

But now suppose we cut a slit in the opposite direction, Fig. 310, then the small squares would be unable to resist the tension across the cut section, and the resistance of the tube would be utterly destroyed, and if

a series of these slits be cut the tube would on the application of T open out into disconnected ribbons.

It is not, however, necessary to cut the slits across $1b$, etc., to destroy the tube; for if we cut the slits longitudinally as aE, 1F (Fig. 308), then there can be no resistance offered to the shear along ab, 12, etc., and hence the tube is destroyed just as much as if we cut it through KK.

[The question just considered has a practical bearing in determining the relative importance of flaws in a shaft. If the shaft always rotate in one direction, it appears that a flaw lying in one direction will hardly weaken the shaft, while one at right angles entirely destroys the resisting power of the space it covers. For intermediate directions—*e.g.* longitudinally—although theoretically there is no resisting power, yet if the sides be held together by the rest of the metal, friction may be developed sufficient to withstand the shear. It would not of course do to rely on such actions as this.]

Thick Tube.—Fig. 311 shows two views of a thick

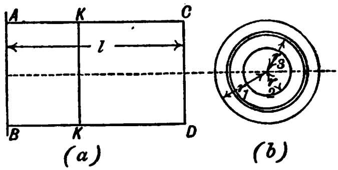


Fig. 311.

tube or hollow shaft; external diameter d_1, or radius r_1; internal diameter d_2, or radius r_2.

This shaft we conceive as made up of a large number of concentric thin tubes, one of which, radius r, small thickness t, is shown in the end view.

If now the shaft be twisted, all these tubes will be twisted, and since they form one solid shaft, we naturally assume that they will all be twisted through the same angle.

Distribution of Stress.—We will now see what distribution of stress will be produced.

Taking the one tube shown. Let

q = shearing stress on it,
l = length of shaft,
i = angle of torsion.

Then

$$i=\frac{ql}{Cr}.$$

But i is constant for all the tubes, hence if q_1 be the stress at the outer surface, and q_2 that at the inner surface, we have

$$\frac{q_1 l}{Cr_1}=\frac{q_2 l}{Cr_2}=\frac{ql}{Cr} \quad \text{(anywhere over the section)},$$

or

$$\frac{q_1}{r_1}=\frac{q_2}{r_2}=\frac{q}{r}.$$

Thus q/r is constant, or $q \propto r$.

If now the moment T be applied to the shaft by forces varying in this way, our assumption above holds good, and the stress is distributed in this manner over all transverse sections. But if not, then near the end the law of distribution will not be $q \propto r$, but it will tend towards this law, and when we get a little way from the end the law will be found to hold good (compare Tension, page 264, and Bending, page 356).

Connection of Stress and Moment.—Let now Fig. 311 (*b*) represent a section through KK in (*a*).

Then KBDK is in equilibrium under T, the twisting moment applied to the end BD, and the stress over the section at KK.

Taking now the ring of radius r,

$$\text{Resisting moment of ring}=2\pi r^2 qt \quad \text{(page 402)},$$

and

$$\frac{q}{r}=\frac{q_1}{r_1},$$

where q_1 is the external, and therefore the greatest, stress.

$$\therefore \text{ Resisting moment of ring} = 2\pi r^2 t \times \frac{q_1}{r_1} . r,$$
$$= \frac{q_1}{r_1}(2\pi r t \times r^2).$$

But $2\pi r t \times r^2$ is the area of the ring multiplied by the square of its radius; that is, it is the polar moment of inertia of the ring about the centre of the section.

$$\therefore \text{ Resisting moment of ring} = \frac{q_1}{r_1} \times \text{I of ring.}$$

And hence, adding the resisting moments of all the tubes together,

$$\begin{matrix}\text{Total resisting moment}\\ \text{of section}\end{matrix} = \frac{q_1}{r_1} \text{ (sum of I's of rings)},$$
$$= \frac{q_1}{r_1} \mathrm{I},$$

where I represents the polar moment of inertia of the section about its centre. Hence the greatest stress on the section is given by

$$\frac{q_1}{r_1} \mathrm{I} = \mathrm{T},$$

since the resisting moment of the section balances T.

If the stress at any other point be required we find it from

$$\frac{q}{r} = \frac{q_1}{r_1}.$$

For the value of I we have

$$\begin{aligned}\mathrm{I} &= \text{I of circle of radius } r_1 - \text{I of circle of radius } r_2,\\ &= \tfrac{1}{2}\mathrm{A}_1 r_1^2 - \tfrac{1}{2}\mathrm{A}_2 r_2^2 \quad \text{(page 214)},\\ &= \tfrac{1}{2}\pi(r_1^4 - r_2^4), \quad \text{or } \frac{\pi}{32}(d_1^4 - d_2^4),\end{aligned}$$
$$\therefore \mathrm{T} = \frac{q_1}{r_1} \times \frac{\pi}{2}(r_1^4 - r_2^4),$$
$$= \frac{\pi}{2} q_1 \frac{r_1^4 - r_2^4}{r_1}, \quad \text{or } \frac{\pi}{16} q_1 \frac{d_1^4 - d_2^4}{d_1}.$$

If the shaft be solid then r_2 or $d_2 = 0$, and

$$T = \frac{\pi}{2} q_1 r_1^3, \quad \text{or } \frac{\pi}{16} q_1 d_1^4.$$

We may now drop the suffixes, remembering that q is the greatest stress and d the outer diameter, and we obtain

$$T = \frac{\pi}{16} q d^3,$$

the usual formula. For the hollow shaft the suffixes should be retained.

Comparison of Solid and Hollow Shafts.—Comparing the formulæ

$$\frac{p}{y} = \frac{M}{I}, \quad \text{and } \frac{q}{r} = \frac{T}{I},$$

we see that the same remarks which applied in bending, as to removing the bulk of the metal away from the axis as far as possible (page 372), will also apply in the present case. The present is, however, simpler, because, the section being always circular, we have only one way of removing the metal away from the axis, viz. to make the shaft hollow, and there is no question similar to the ratio of flanges to web, etc., in bending.

To see what we gain by making a shaft hollow, let us compare together the resisting powers of two shafts of equal sectional area, and therefore equal weights, one solid the other hollow. Let

d = diameter of solid shaft,
d_1 = external diameter of hollow shaft,
d_2 = internal " " ",
f = greatest stress allowed
(so q at the outer edge is to be f),
T = resisting moment of solid,
T' = " " hollow.

Then

$$T = \frac{\pi}{16} f d^3,$$

$$T' = \frac{\pi}{16} f . \frac{d_1^4 - d_2^4}{d_1},$$
$$\therefore \frac{T'}{T} = \frac{d_1^4 - d_2^4}{d_1 . d^3}.$$

Also the sectional areas being equal,

$$\frac{\pi}{4} d^2 = \frac{\pi}{4}(d_1^2 - d_2^2),$$

whence

$$\frac{T'}{T} = \frac{d_1^2 + d_2^2}{d_1 d} = \frac{d_1^2 + d_2^2}{d_1 \sqrt{d_1^2 - d_2^2}},$$
$$= \frac{1 + \left(\frac{d_2}{d_1}\right)^2}{\sqrt{1 - \left(\frac{d_2}{d_1}\right)^2}}.$$

So that the nearer d_2 approaches to d_1 the greater is the gain, the limiting value being 2; but for this $d_2 = d_1$, or the thickness is zero, hence the diameter must be infinite.

In actual practice the thickness $d_1 - d_2$ is limited by considerations of local stiffness, similar to those limiting the thinness of flanges and webs (page 373). Hence the ratio $d_2 : d_1$ is limited. Taking the value 1 : 2 we get

$$\frac{T'}{T} = \frac{1 + \frac{1}{4}}{\sqrt{\frac{3}{4}}} = 1.44,$$

so the shaft is nearly half as strong again. (For the same elastic strength, see note at end of ch. xxi.)

Equivalent Solid Shaft.—It is for many purposes convenient to express the strength of a hollow shaft by giving the diameter of a solid shaft of the same material which would be equally strong; this is called the equivalent solid diameter. Let

d_1 = actual external diameter,
d_2 = ,, internal ,,
d = equivalent solid diameter,
f = stress allowed.

Then

$$\text{Resisting moment of actual shaft} = \frac{\pi}{16} f \frac{d_1^4 - d_2^4}{d_1},$$

and of equivalent solid $= \frac{\pi}{16} f . d^3$. Whence

$$\frac{d_1^4 - d_2^4}{d_1} = d^3,$$

and

$$d = \sqrt[3]{\frac{d_1^4 - d_2^4}{d_1}} = d_1 \sqrt[3]{1 - \left(\frac{d_2}{d_1}\right)^4}.$$

Diameter of Shafting to Transmit a given Horse Power.—An engine of given I. H. P. runs at N revolutions per minute; it is required to find the necessary diameter of its shafting. First let

T_m = mean twisting moment of engine in tons-inches.

Then

$$\text{Work done per revolution} = T_m \times 2\pi \text{ inch-tons.}$$

But

$$\text{Energy exerted per revolution} = \frac{\text{IHP} \times 33000}{\text{N}} \text{ ft.-lbs.},$$

$$= \frac{\text{IHP} \times 33000 \times 12}{\text{N} \times 2240} \text{ inch-tons.}$$

Whence, equating energy and work, we obtain

$$T_m = \frac{\text{IHP}}{\text{N}} \times \frac{33000 \times 12}{2240 \times 2\pi}.$$

But in chap. ix. we proved that the twisting moment does not remain constant at its mean value, but varies; the ratio of maximum to mean twisting moment depending partly on considerations there discussed, viz. number of cylinders, and connecting-rod ÷ crank ratio, and partly on other considerations, into which we cannot enter. The effect is then that if

$$T = \text{greatest twisting moment,}$$

we have

$$T = KT_m,$$

where K is a constant greater than 1, depending on the foregoing considerations. A mean value for a pair of cylinders would be about $1\frac{1}{2}$ for the propeller shafting and 2 for the crank shaft. The greater value in the latter case includes an allowance for the very severe bending to which a crank shaft is subject.

But the shaft must be designed to be strong enough to withstand T, whence taking a solid shaft

$$\frac{\pi}{16}fd^3 = \mathrm{T} = \mathrm{KT}_m,$$
$$= \frac{\mathrm{K.IHP}}{\mathrm{N}} \times \frac{33000 \times 12}{2240 \times 2\pi},$$

whence

$$d = 5.3\sqrt[3]{\frac{\mathrm{K}}{f} \cdot \frac{\mathrm{IHP}}{\mathrm{N}}} \text{ very nearly } (f \text{ in tons}),$$

which gives the actual diameter in inches of a solid shaft, or the equivalent solid diameter if we use a hollow one.

In the latter case we have

$$d_1\sqrt[3]{1 - \left(\frac{d_2}{d_1}\right)^4} = d,$$

whence, being given the ratio d_2/d_1, we obtain the values of d_2 and d_1. Usual values of the ratio are from $\frac{1}{2}$ to $\frac{5}{8}$. Or we may be given that $d_2 - d_1$ is not to be less than a certain thickness, say for large shafts about 3 ins.; then, using this value, we obtain the two diameters.

EXAMPLES.

1. If the greatest shearing stress allowed on the pin of a pin joint be $\frac{4}{5}$ of the tensile stress allowed in the metal of the rods joined, show that the diameter of the pin should very approximately equal the diameter of the rod.

2. A single riveted lap joint in $\frac{1}{2}$ inch plate is subject to a load of 3 tons per square inch of the plate section through the line of rivets. The rivets are $\frac{3}{4}$ in. diameter, pitch $1\frac{7}{8}$ in. Find the shearing stress on the rivets, and the efficiency of the joint.

Ans. 3.8 tons; .6.

3. The steel plates of a girder are $\frac{3}{4}$ in. thick, riveted with 1 inch rivets. The joint is treble riveted, double butt strap, and the shearing strength of the rivets is $\frac{4}{5}$ the tensile strength of the plates. Find the pitch. *Ans.* $6\frac{1}{4}$ ins.

4. A square bar of steel is under a tensile pull of 4 tons per sq. inch along its axis, and a compressive stress of 2 tons at right angles to the axis. Find the direction of a plane on which there is a pure shearing stress, and its amount.

Ans. At an angle $\tan^{-1}\frac{1}{\sqrt{2}}$ with the axis.

5. Find the greatest twisting moment a steel tube 12 ins. mean diameter, $\frac{1}{8}$ inch thick, can withstand, shearing stress allowed 4 tons per sq. inch. If the length be 6 ft. find the angle of torsion. *Ans.* 113 tons-ins.; 5.86°.

6. Find the diameter of a solid steel shaft to transmit 6000 H. P. at 116 revolutions per minute, the maximum twisting moment being 1.3 times the mean, and stress allowed 10,000 lbs. per sq. inch. *Ans.* $12\frac{1}{4}$ ins.

7. Find the size of a hollow shaft to replace the preceding, diameter of hole $\frac{5}{8}$ the outside diameter. Estimate the saving in weight in 50 feet of shafting.

Ans. $12\frac{7}{8}$ ins. outside, 8 ins. inside diameter; 8170 lbs.

8. The angle of torsion of a cylindrical shaft is required not to exceed one degree for each 5 feet of length, and the stress not to be greater than 12,000 lbs. per sq. inch. Determine the diameter of shaft above which the second condition, and below which the first condition, fixes a limit to the greatest twisting moment which may be applied.

Ans. 8 inches; a T. M. of 540 tons-inches then produces both the limiting torsion and stress. Below this the torsion at 12,000 lbs. stress would be more than allowed, and above this the stress at the given torsion would be more than 12,000 lbs.

9. If the modulus of rigidity be 4800 in ton-inch units, what is the greatest stress to which the material of a shaft should be subjected, in order that the angle of torsion may not exceed one degree for each length of ten diameters.

Ans. 4.2 tons per sq. inch.

10. In renewing the engines of a ship, the speed of revolution is increased by one-third, the horse power is doubled, the ratio of maximum to mean crank effort is altered from 1.5 to 1.25, and the strength of the material used for the shaft is greater by 25 per cent. Show that the size of the shaft is unaltered.

CHAPTER XXI

EXPERIMENTAL FACTS—ELASTICITY—STRENGTH—RESISTANCE TO IMPACT

IN all the preceding chapters we have assumed that materials obey certain laws connecting together the stress in a piece and the alteration of form produced. The laws we have assumed are

$$p=E\frac{x}{l}, \quad \text{or } Ee,$$

and

$$q=C\phi,$$

which may both be, in words, stated as follows:—

Stress varies directly as the corresponding strain.

These laws are the result of experiment, and they are satisfied, within certain limits, by the principal materials we have to deal with, allowing for the small irregularities which we always find in actual practice.

Elastic State.—In most materials, if stresses less than a certain amount be applied to a piece, it is found that the laws above are satisfied; and that, when the stress is removed, the piece returns exactly to its original condition. The material is then said to be perfectly elastic or in the elastic state; by elasticity being meant the power of resuming its original shape and size.

Proof Stress.—If, however, a certain stress depending on the nature of the material be reached, it will be found that, when released, the piece no longer

resumes its original dimensions, but a permanent alteration of form has taken place, or there is a Permanent Set. If we apply stresses greater than the above, it is found that the simple laws connecting stress and strain no longer hold. We say then that the elastic state is passed; and the limiting stress, above which these changes occur, is called the Elastic Limit or Proof Stress.

It must not be supposed that there is a sharply defined limit always the same in all cases for the same metal or material—in fact in some materials, cast-iron for instance, there is, as we shall see, no perfectly elastic state; yet in most cases a limiting or proof stress may be found, below which the material is *practically* perfectly elastic, and above which it deviates entirely from perfect elasticity.

Now in all practical cases we want a piece of material not only not to break, but also to keep its dimensions unchanged, omitting the elastic stretching or change of shape necessarily accompanying stress, and which disappears when the stress is removed. Hence the examination of material in this state is of the most practical importance, which explains why the whole of our preceding work has been confined to this. It is however necessary, for the purpose of determining, for one thing, the elastic limit, and also for investigating the manner of resistance to certain actions not hitherto considered, that we should examine how material behaves under stresses of any magnitude up to those which actually break it. We are now therefore going to describe what is actually found by experiment to occur; the results have been partially anticipated by what we have just been saying, but the preliminary statement which has been made will be found, probably, useful, by showing beforehand the principal points which have to be considered.

Testing Machines.—When a piece of material is tested the load should be applied gradually, for reasons

which will hereafter appear, and the change of form which takes place should be continually measured, while the corresponding load is noted. The loads to be dealt with are large, so that if they be applied, for example, in tension, by hanging weights directly on to a rod, very heavy weights have to be manipulated. Again, the changes of length produced are, especially inside the elastic limit, very small, so that if they are to be measured by ordinary means, the bar experimented on would need to be very long. In an experiment by Hodgkinson, a bar 50 ft. long of $\frac{1}{2}''$ diameter was used, and the bar was loaded by successive loads of 5 cwt. being placed in the scale pan at one end.

Testing is now so much resorted to—not for scientific purposes so much as for commercial ones, very little material being now bought without tests being specified for—that special machines, in which the application and measurement of the loads can be easily carried out on specimen pieces of moderate size, are used. These machines are principally fitted for tension experiments, these being the chief experiments carried out; but they can be also used for applying compression, bending, or torsion, by means of special fittings. We cannot here enter into the details of fittings, but must confine ourselves to a simple explanation of the principle on which the majority of these machines act.

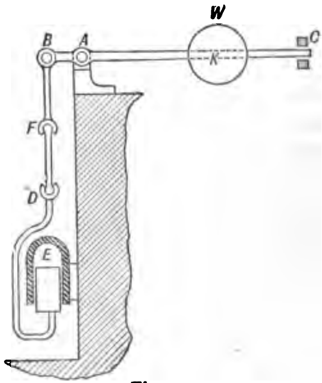


Fig. 312.

BAC is a lever supported on a knife edge at A, FD

is the piece to be tested connected by a shackle on top to the point B of the lever, and by a shackle at the bottom to the ram of a hydraulic press E. The tension is applied by the hydraulic press, so that it is applied steadily, while the lever is kept balanced between two stops, shown at the end C, by running out the weight W by means of a screw which moves it along AC; the position of the jockey weight at any instant also measures the amount of pull, for if P be the pull in the position shown,

$$P \times AB = W \times AK,$$

$$P = W \cdot \frac{AK}{AB}.$$

The distance AB is in the actual machines very small—in the machine used at the Central Institute by Professor Unwin it is 4 ins., the length AC being 200 ins.—so that W need not be a very heavy weight to produce a heavy pull. Thus in the above case the jockey weight of 1 ton can produce a pull of 50 tons. If the greatest pull of the jockey weight is not sufficient, then the end C can be initially loaded before the jockey is run out.*

We now have the load applied and measured, and we require next to measure the extensions produced. For this purpose the specimen piece should have a fairly long part of constant sectional area, and which will thus extend evenly (this we shall also see is necessary for another reason); then if two points be marked on this, their distance apart can be measured by some special measuring instrument, and so the extensions of this given length recorded.

Tension — Graphic Representation — Stress-Strain Curve.—A test having been made, the loads and the corresponding extensions can be written down

* Fig. 312 is the diagrammatic arrangement given by Professor Unwin in his book, *The Testing of Materials of Construction*, where full descriptions of all varieties of testing machines are given.

in tabular form; but the nature of the law connecting them is not shown clearly by tables, and so recourse is had to a graphic representation: the extensions are marked off as abscissæ, and the corresponding loads as ordinates, then a curve drawn through the tops of the ordinates is called the Curve of Stress and Strain.

We will first describe the general nature of such curves, and afterwards give particular values (page 435). The figures are not drawn to scale.

Fig. 313 shows the shape of the curve for a material such as wrought-iron or mild steel.

Initial Strains.—We must first notice that when a bar of metal is tested for the first time the extensions are

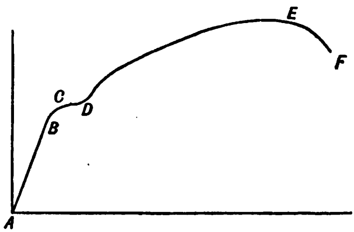


Fig. 313.

more or less irregular, and permanent set may be produced with small loads. This effect is believed to be due to irregularities or initial strains caused during the manufacture, since the irregularities nearly disappear after the bar has been stretched by a moderate load and released.

Elastic State.—Commencing now to consider the curve, we see that the portion AB is straight, *i.e.* the extension varies as the load, and also it will be found that in this state practically no permanent set is produced. The bar is then in the elastic state.

After passing B the curve bends slightly away from the straight line, so the piece is not now perfectly elastic, but the deviation is small.

Breaking-down Stress—Proof Stress.—We now reach the point C, and here a sudden change occurs, the piece suddenly elongating to a considerable amount (comparatively), so that CD is parallel to the axis. This point is always well marked, and can be easily detected by the outer scale on the bar becoming detached. Now it is difficult to decide exactly what is the true limit of elasticity or proof stress; but there is no doubt that a marked change in the properties of the bar takes place at C, a decided permanent set being now produced; hence C is called the breaking-down point.

We now go on increasing the load and corresponding extensions, and also, if we remove the load, we find we increase the permanent set or elongation, until the point E is reached, where the load reaches a maximum; but after passing E up to F, where rupture takes place, we have the apparent anomaly of increasing length caused by a decreasing load.

Change in Cross-Section.—In order to explain the point just mentioned we must go back and inquire how the cross dimensions have altered, if at all. We shall find that the following changes have occurred. Up to the point B, both length and cross-section have altered very little; but after passing B the increasing extension has been accompanied by a decrease in cross-section, this decrease continuing fairly uniform over the length of the plane part of the piece, as shown exaggerated in Fig. 314 by the dotted lines.

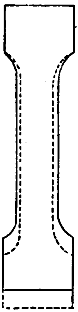


Fig. 314.

This holds true up to the point E, and here it is found that some point, being weaker than the remainder, begins to draw out more rapidly, so that the piece assumes the shape of Fig. 315.

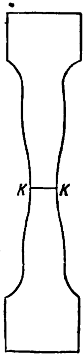


Fig. 315.

Now the diagram shows the connection between *Load* and strain, not *Stress* and strain; but what determines the question of drawing out is not load but stress; and the section at KK becoming reduced below the original section, the load may and in fact does produce on it an increasing stress as the section diminishes, even although it (the load) be also diminishing. For example, suppose the ordinate at E $\frac{4}{3}$ that at F, while the section KK meanwhile diminishes to $\frac{3}{5}$ of its size at E. Then

$$\text{Stress on section at E} = \frac{\text{Load at E}}{\text{Section at E}}.$$

$$\text{Stress on section at F} = \frac{\text{Load at F}}{\text{Section at F}},$$

$$= \frac{\frac{3}{4}\text{ load at E}}{\frac{3}{5}\text{ section at E}},$$

so that, although the load is reduced, the stress per sq. inch of the section is increased in the ratio 5 to 4.

Apparent and Real Ultimate Stress.—It is clear that any large deformation of the parts of a machine or structure is practically inadmissible, and we are therefore certain that the proof stress must be less than the stress at C, the breaking-down point, though it may in reality be much below this limit. But it is also of interest to know what stress actually breaks the material.

Now the breaking load usually found will be the ordinate at E, say P_E tons, and if A be the original sectional area of the piece in sq. ins., then the ultimate or breaking stress is said to be P_E/A tons per square inch. But this is not the real stress across the section which broke, for if that section be A', the stress at the instant of breaking was P_F/A' tons per sq. inch. It is often said that P_E/A is the Apparent Ultimate Strength of the metal, and P_E/A' the Real Ultimate Strength. It does not, however, follow that this so-called real is a better measure than the apparent strength. For suppose

a 1 inch square bar broke under a load of 26 tons, say, while the broken section was only .6 sq. ins. Then the Real Tenacity, as it is sometimes called, is, when thus estimated,

$$\frac{26}{\cdot 6}=43\tfrac{1}{3} \text{ tons per sq. in.}$$

But the real load (P_F) at the instant of fracture was much less than 26 tons (P_E), say only 21 tons, and the real stress only 21/.6 or 35 tons per sq. in. And so it is perhaps on the whole better to say the tenacity or ultimate strength is 26 than $43\frac{1}{3}$ tons. There are objections even to regarding 35 tons as the true tenacity.

Shape of Specimens.—We said (page 416) that there is a reason for having a fairly long piece of uniform section other than for measurements of extension, and this reason is connected with the question we have just been considering. Suppose, instead of having a long straight part, we make a groove in the specimen piece, as Fig. 316, then we have practically no length at all for extension to take place in; and if we experiment on such a piece we find that it does not draw out at all, but finally breaks across at KK, the broken section being very little less than the original one, *i.e.* the original one at KK; the groove is now the bar, the other parts being its ends. Also the breaking load will be found to be much higher than for a bar of section KK throughout; and consequently the apparent tenacity of the metal will be higher than it would be if obtained by breaking a specimen of the ordinary dimensions. The real tenacity will, however, be found to be much the same as for the ordinary specimen, the whole of the difference in the results appearing to be due to the difference in the broken sections. Now it seems clear that a grooved bar cannot really be stronger than one of uniform section KK—in fact we shall show that for certain kinds of load the grooved bar is actually the

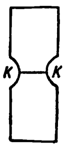


Fig. 316.

weaker,—consequently we say, either that the true tenacity should be taken as a measure of the actual strength, to which we have seen there are objections, or better, that specimen pieces should have fairly long parts of uniform sectional area, as we have already stated they now always have.

Plastic State—Flow of Metals.—Returning to the consideration of the curve ABCDEF (Fig. 313), we see that up to E it is necessary, in order to produce extension, to continually increase the load and the stress. But after E it appears that a considerable amount of extension can be produced with a decreasing load, which may cause, as we have seen, a constant or slowly increasing stress on the weak section. The metal has now properties directly opposed to its elastic properties. For within the elastic limit strain is proportional to stress, whereas in the present case strain is nearly independent of stress.

There are certain materials, *e.g.* clay, which are practically always in this latter state, for if we press a piece of clay we can change its shape, *i.e.* produce strain, to any extent, by keeping up the pressure long enough, without increasing it at all. Materials of this kind are called Plastic, and a perfectly plastic material would be one in which a given stress could produce any amount of strain, depending only on the time of application. Since the metal appears during EF to be nearly in this condition, we say it is in the plastic state. The change of form produced in the plastic state is often called Flow, and the effect is called the Flow of Metals. This effect is of importance in many industrial occupations, one case which we may mention being the operation of wire-drawing. Here a bar of metal is pulled through a hole of smaller section than the bar, and the metal flows, forming a smaller bar; this process being repeated with continually diminishing sizes of hole, we finally obtain a wire. The effect is

complicated by the pressure of the sides of the hole, and so is not a case of simple tension but is similar somewhat to that shown in Fig. 300. The manufacture of lead pipes, or solid-drawn copper pipes, depends on the same phenomenon.

Effect of Stress beyond the Elastic Limit.—When a bar is subject to stress within the elastic limit, on removal of the stress the bar returns to its original form, and retains all its original qualities. But if the stress be increased beyond the elastic limit and then removed, the bar in the first place does not return to its original length, but has a permanent increase of length or set; and secondly its properties are altered.

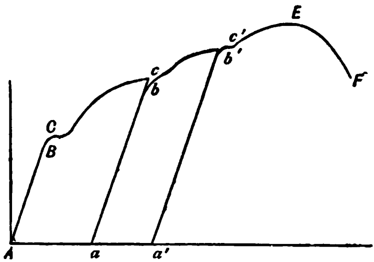


Fig. 317.

Let a bar be stretched, its stress-strain diagram being as in Fig. 317, till the point *b* is reached. Remove the load, then the bar contracts, and the relation between load and strain is shown by the line *ba*; A*a* shows the amount of permanent set, and *ba* is almost exactly parallel to AB.

Now let the bar be stretched again, when it will extend according to the line *ab*, and will remain perfectly elastic not only up to the same elastic limit as it originally had, but right up to the point *b*, and we say the elastic limit is raised up to the stress which was origin-

ally applied. There will now be a period of imperfect elasticity *bc*, a new breaking-down point *c*, and so on, as shown in the figure. The effect can be repeated any number of times, so long as the point E of Fig. 313 has not been reached.

Since the elastic limit of a bar can be increased in the above way, it may be thought that the bar is strengthened, and so in a limited sense it is; but for practical use it is not strengthened but weakened, for reasons which we shall now see.

Effect of Impact.—We have so far supposed pieces of material to be gradually loaded, but in very many cases the loads which they have to bear in practice will be applied to them suddenly, or may be brought against them with a certain velocity. We will now see what effect this will have.

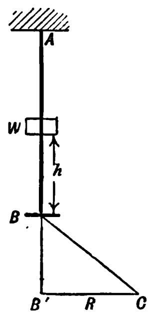


Fig. 318.

AB is a rod fastened at A, a weight W encircles the rod, and is let fall from a height *h* on a collar at B. AB will then stretch to a length AB′, shown exaggerated; if this be the utmost extension the weight W will then be at rest.

Consider now the period from the moment W was let go until it stops at B′. Then

$$\text{Initial K. E.} = 0,$$

and

$$\text{Final K. E.} = 0.$$

$$\therefore \underset{\text{(by gravity)}}{\text{Energy exerted}} = \underset{\text{(against resistance of rod)}}{\text{work done}}$$

We will now consider two cases:—

Case I.—When W and *h* are such that the rod is not stretched beyond the elastic limit.

Let R be the final resistance of the bar, *i.e.* when it is stretched to AB′.

Then, setting off B′C = R, and joining BC, BC will be

the curve of resistance, since within the elastic limit load varies as strain, and when the load is applied gradually, as in the testing machine, there being no K. E. developed, the load and resistance at every instant balance, therefore resistance varies as strain.

$$\therefore \text{ Work done} = \text{area of } BB'C,$$
$$= \tfrac{1}{2}R \times BB'.$$

But if A be the sectional area of the bar,

$$\frac{R}{A} = \text{final stress} = E\frac{x}{l},$$
$$= E.\frac{BB'}{AB}.$$
$$\therefore \text{ Work done} = \tfrac{1}{2}R \times \frac{R}{A} \times \frac{AB}{E},$$
$$= \tfrac{1}{2}\frac{R^2}{A \times E}.AB.$$

And

$$\text{Energy exerted} = W(h + BB'),$$
$$= W\left(h + \frac{R}{A \times E}.AB\right).$$
$$\therefore \; W\left(h + \frac{R}{A \times E}.AB\right) = \tfrac{1}{2}\frac{R^2}{A \times E}.AB,$$

an equation which will give us the value of R, and therefore of R/A, the final stress produced, when W and h are given.

[Compare with the foregoing chap. xiii. page 271.]

One case gives a simple result, viz. when $h = 0$, so that the load is *suddenly* applied, although not with any initial velocity. This gives

$$W.\frac{R}{A \times E}.AB = \tfrac{1}{2}\frac{R^2}{A \times E}.AB,$$

or

$$R = 2W,$$

so that by sudden application the effect of a load is doubled, and this explains why it is necessary when

testing to apply loads gradually and not suddenly. Also we see that if we do not wish a material to be strained beyond the elastic limit, we must not apply suddenly to it loads which would produce, if applied gradually, more than half the proof stress.

In many cases we can simplify our preceding work by omitting from the energy exerted the term W.BB′, for BB′ is a small quantity. The equation then becomes simply

$$Wh=\tfrac{1}{2}Rx \qquad \text{(putting } x \text{ for BB}').$$

Also the impact may be effected on a bar lying horizontally, the weight W being by some means given a velocity V f.s. say.

In this case

$$\text{Initial K. E.}=\frac{WV^2}{2g},$$
$$\text{Final K. E.}=0.$$
$$\text{Energy exerted}=0,$$

whence

$$0=\tfrac{1}{2}Rx+0-\frac{WV^2}{2g},$$
$$\therefore\ Rx=\frac{WV^2}{g},$$

a form which is often used for such cases as a blow from a hammer, even where the rod is vertical, because x is negligible. The equation is of course identical with

$$\frac{Rx}{2}=Wh, \quad \text{since } \frac{V^2}{2g}=h,$$

$WV^2/2g$ is called the Energy of the Blow.

In chap. xiii. we have calculated the resilience of a bar, or the greatest amount of work which can be done on it without exceeding the proof stress, and from our present work we see that we may now define the resilience as the greatest amount of energy which can be applied to a bar without injuring it, or at least altering its properties.

A very convenient way of stating the resilience of a material is by giving the height from which a piece may fall without injury.

For this we have simply

$$Wh = \text{resilience} = \frac{f^2}{2E} \times \text{volume},$$

W being the weight of the piece itself.

$$\therefore \; W = w \times \text{volume},$$

where w is the weight of unit volume.

$$\therefore \; h = \frac{f^2}{2Ew} \quad (h \text{ is called the height due to the resilience}).$$

This will be found on calculation to be for most materials very small, *e.g.* for wrought-iron it is about 20 ins. only. We see then that when such materials withstand blows they will generally be strained beyond the limit of elasticity. We proceed then to examine

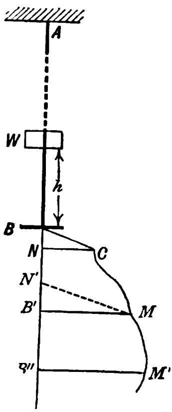


Fig. 319.

Case II.—The resistance now can only be shown graphically, since it follows no law which can be expressed algebraically. AB is the bar, broken in the figure, since its length must be considerable compared with the extensions. Produce AB, and draw the load-strain diagram BCM′M. BC is very nearly vertical if drawn to any ordinary scale. The slope which we have given it in former figures was for clearness. Let the blow be such as to cause the extension BB′. Then B′M is the equivalent steady load which produces the same stress as the actual blow, and we have from the principle of work

$$\begin{aligned} \text{Energy of blow} &= \text{work done}, \\ &= \text{area BCMB}'. \end{aligned}$$

Now of this area the resistance inside the elastic limit only provided the very small portion BCN, and hence we see that materials having curves, such as we have here drawn, resist impact chiefly by being strained above the elastic limit.

Ductility.—The materials of which we have just spoken have the property of extending considerably before breaking, to which property, when combined with tenacity, the name Ductility is applied, and the materials are called Ductile Materials. Wrought-iron, mild steel, and copper are the chief examples.

Effect of raising Limit of Elasticity on Resistance to Blows.—We can now explain the statement on page 423, relative to the effect of stress beyond the elastic limit having in many cases a weakening effect. For in Fig. 319 we have stress beyond the elastic limit, and consequently the load-strain curve of the bar will now be, as shown, dotted. Let now a second blow of equal energy be struck, then the bar must extend to B″ such that MM′B″B = BCMB′, and the effect produced is equivalent to the application of a steady load M′B″. In the figure we have taken such dimensions that M′ is the point of maximum load and development of local contraction, and it consequently follows that the second blow has practically destroyed the bar, owing to the raising of the elastic limit caused by the first blow. The effect here shown also explains how by continued blows a piece of material will be finally broken, although it may be able to withstand one, two, or even a large number.

Coefficients of Strength—Factors of Safety.—In former times, when the strength of materials had not been so thoroughly investigated, and testing machines were rare, very little about materials beyond their ultimate strength was known, and this strength being given, the working stress allowed was taken less than this in a certain ratio, called a Factor of Safety. This factor was taken

empirically, and depended on the nature of the loads to which the piece was subject, being usually about 6 to 8. The method still holds to a very large extent, but with better knowledge of the strengths its value is not determined so purely empirically, and also the true nature of the ratio is better understood.

The name factor of safety is a bad one, since it gives the idea that, when it is, say, 6 for a given bar, the bar is for safety 6 times stronger than it need be. But now take the case, say, of a wrought-iron bar, ultimate strength 54,000 lbs. per sq. inch (apparent is always understood); then the working strength is 9000 lbs., and a bar which was to be exposed to a working load of 9000 lbs. would be 1 sq. in. in sectional area, but would not be 6 times stronger than necessary. For probably the load would be applied suddenly—as in a piston-rod at each end of the stroke—and will thus stress the bar up to 18,000 lbs. per sq. in.; and this 18,000 will be about the limit of elasticity, which, as we have seen, is the point beyond which the metal is injured. Hence, then, in this case, instead of being six times too strong, the bar is only barely strong enough; because a continuous repetition of suddenly applied loads a little above 9000 lbs. would in time break it. If, however, the load 9000 lbs. were to be applied once for all and to remain constant, then the bar would be twice as big as necessary, and there would be a true factor of safety of 2.

The ratio of working to ultimate strength should then depend on the ratio of proof to ultimate, and to the manner in which the load is applied. It is generally 6 for piston-rods and similar pieces, or perhaps a little greater to allow a small real safety factor. For bridges in which most of the load is constant but some is variable, *i.e.* the loads which cross, 5 is the usual value; while for the shell plates of a boiler, subject to a gradually applied steady steam pressure, 4 and slightly under is allowed.

It is, however, probable that in the future proof stress will be the determining factor and not ultimate stress.

Compression of Ductile Material.—We have taken the case of tension at great length, because it is the case of which most is known, and the results are most clearly defined. All the work as to the effect of impact of course applies equally to all cases, also the effect of stress beyond the elastic limit, so we shall not need to consider other cases so fully.

In the present case, referring to chap. xiii., there is a difference between the results obtained from long pieces and those from short ones. The former being the more important we will first consider them.

Long Pillars—Gordon's Formula—All Materials.—When a long pillar is loaded, as Fig. 320, then, if the load were right in the axis and the pillar perfectly straight and homogeneous, it would remain straight, and the stress on the section would be W/A, A = area of section.

Actually, however, these conditions are never fulfilled, and thus the pillar bends, and as the load increases the bending also increases, and the pillar finally gives way by the combined effects of compression and bending.

We cannot here enter into the analysis of these effects, but must simply give the empirical formula constructed by **Gordon** and modified by **Rankine** to represent the results of an extensive series of experiments by **Hodgkinson.**

This formula gives the breaking load W for a pillar of length l as follows :—

$$W = \frac{fA}{1 + \frac{l^2}{cnh^2}},$$

A = sectional area, f = coefficient of strength.

n is the constant in the formula $I = nAh^2$ (page 368), h is depth of the pillar in the plane of bending, *i.e.* for a

circular pillar $h = d$, and for a rectangular pillar h = least breadth, since that is the direction the pillar will evidently bend in.

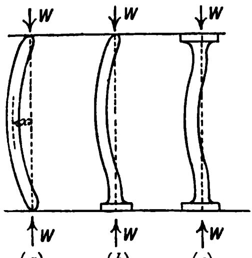


Fig. 320.

c is a constant depending on the manner of connection of the ends of the pillar, which we can see will affect the bending, for, looking at Fig. 320, in (a) we have a pillar with both ends free to turn, and it bends accordingly into one large arc, the bending moment at the centre being Wx. Then in (b) one end of the pillar cannot turn, it bends into two arcs, and Wx the greatest B. M. is less, since x is less. The decrease is still more in (c), where neither end can rotate, x being less still.

The values of f and c are as follows :—

	f	c		
		(a)	(b)	(c)
Wrought-iron	36,000	9,000	18,000	36,000
Steel (mild)	42,000	9,000	18,000	36,000
Cast-iron	80,000	1,600	3,200	6,400
Dry Timber	7,200	750	1,500	3,000

We have here also given the results for cast-iron and timber, which are not ductile; but the formula applies to these as well as to the ductile ones; and it is the practical formula which in nearly every case is used, because pieces so short as not to bend at all are not often used compared with the longer pieces.

Working Strength of Pillars.—The value of W given by Gordon's formula is the crushing load, and

hence in calculating the size of a pillar for a given load a "factor of safety" must be allowed. This is a case in which accidental circumstances may produce large stresses, *e.g.* a pillar may be a little bent when set up, and then there is at once a bending moment produced at the bent part, hence the factor of safety should be large, and it is usually taken at 8 or more. To find then the size of a pillar to carry a load of 10 tons say, we should put W at least 80 tons.

Crushing of Short Pieces of Ductile Material.—When the length of a piece is not more than about $1\frac{1}{2}$ times its diameter, it will not give way by bending but in a manner we will explain.

In the first case, the elastic limit and modulus of elasticity are practically the same as for tension.

When the elastic limit is passed, then we have permanent set, and an *increase* of cross-sections, as opposed to the *decrease* in tension; and this increases with increasing loads, the stress per sq. inch on the section also increasing up to a certain point (see page 419).

After a certain stress is reached the load increasing produces an increase of section such that the stress remains fairly constant (page 419). And finally, when a certain compression is reached, the metal gives way by cracking round the circumference, as Fig. 321, which represents the compression of a block of steel experimented on by Sir W. Fairbairn.

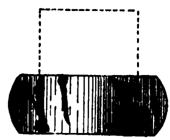


Fig. 321.

There is nothing corresponding to the local contraction in tension; the bulging of the block in the middle is not due to any local cause, but partly to the friction of the pieces by which the pressure is applied holding the ends together, and partly probably to another cause somewhat outside our present scope.

The curve of stress and strain is very similar to that for tension, but the results are not so definite. The

ultimate strength appears to be somewhat less than for tension; but since giving way is due to the formation of cracks, there is an accidental character which causes in some cases considerable discrepancies. This, however, is not of much consequence, because *elastic* strength is what we want in practice.

Rigid Materials—Tension.—Rigid materials are those possessing opposite characters to ductile materials; so they can neither be drawn out nor hammered out, but break before any appreciable change of shape can be produced. A typical example is cast-iron.

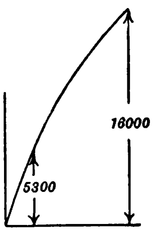


Fig. 322.

Fig. 322 shows the shape of the stress-strain curve, and it differs, we see, entirely from the case of ductile material. In the first place there is no real elastic state at all, for permanent set is produced by all stresses except very small ones. Also, except for very small stresses, we have not

$$p = Ee,$$

but

$$p = Ee\,(1 - ke) \quad (k \text{ being a constant average value } 209).$$

Against this, however, we set the fact that this imperfectly elastic state continues right up to the breaking point, so there is no breaking-down point and no drawing out locally, the bar breaking across practically the full original area of section. There will be little error, however, in taking cast-iron to be perfectly elastic up to about one-third of the breaking stress, which is about 16,000 lbs. per sq. inch; so below 5300 lbs. per sq. inch the metal may be assumed to be elastic. Cast-iron is very variable in quality, so this can only be taken as an average value; but it is in all cases much weaker in tension than the ductile wrought-iron.

Compression of Rigid Materials.—Long pieces

we have already considered, so far as practical strength is concerned, but there is one point of difference which must be noticed, as compared with ductile material. There is an absence of perfect elasticity, and we have

$$p = Ee(1 - ke),$$

but, moreover, whereas in ductile material E is the same in tension and compression, in rigid material E for compression is less than for tension (see table, page 435). Also the value of k is now about 40 instead of 209.

These results were obtained by Hodgkinson, long bars being compressed, and prevented from bending by enclosure in a frame.

For actual crushing we must have recourse to short blocks, as for ductile materials.

The load-strain curve is then found to be similar in shape to that for tension, but the breaking load is very much higher, being five or six times greater.

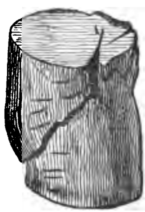


Fig. 323.

The blocks finally give, as shown in Fig. 323, by shearing or splitting on oblique planes.

The characteristics just mentioned are common to all rigid materials, and hence apply to the crushing of stone and of brick. For numerical results see the table at the end of the chapter.

Fibrous Materials — Wood and Ropes.—We have given considerable space to the metals on account of their importance, hence we can only briefly glance at the present kind of materials.

Wood consists of fibres arranged longitudinally, but they are united together loosely; hence wood is strong against tension, the fibres each taking a full share, but can easily be split or sheared longitudinally, and under a compressive load gives way at a comparatively low stress by splitting.

The strength of wood is also affected by its condition; seasoned wood being elastic nearly up to the breaking point, while in green wood the elasticity is imperfect and the strength much less.

Ropes have the fibres arranged spirally, and when under tension, the fibres being pressed together, friction is developed sufficient to prevent their sliding over each other. One effect of this is, however, to weaken each individual fibre, so that a rope is not so strong as the fibres separately would be; nor is a cable so strong as the smaller ropes of which it is made up. The size of ropes is usually expressed by their girth, and hence the breaking load is also expressed in terms of this quantity. Thus

$$T=\frac{C^2}{k},$$

where

T = breaking load in tons,
C = girth in inches,
k = constant.

k depends on the material, and also somewhat on the size, small ropes being, as we have seen, stronger comparatively. Thus, for

Hemp, $k=3.3$ ordinary, and a little less for small ropes.
Iron wire, $k=1$, or rather more if $C>6$.
Steel wire, $k=.5$ or even less.

This gives the breaking load, and the "factor of safety" may be 5 for wire ropes and 6 for hemp.

Table of Strength and Weight.—The following table contains numerical values of the various physical constants referred to in Strength of Materials for some of the principal materials used. Working stress is omitted, its determination being explained on page 428, and its value not being constant but varying according to the factor of safety employed.

TABLE of STRENGTH and ELASTICITY in Tons per square inch, and Weight in lbs. per cubic foot.

MATERIAL.	ULTIMATE STRENGTH.			ELASTIC LIMIT.			YOUNG'S MODULUS E.	COEFFICIENT OF DISTORTION C.	WEIGHT.
	Tension.	Compression.	Shearing.	Tension.	Compression.	Shearing.			
ast-iron	7.8	42	..	4.4	9.4	3.5	*6,250	2800	450
Vrought-iron	25	22	22	10.5	10.5	9	13,000	4700	480
teel Boiler Plates	29	..	..	9	9	7	11,500	4200	487
Iild Steel	35	..	..	15.5	..	11.8	13,200	4900	480
empered Steel	..	..	..	85	..	64	16,000	5800	480
opper	15	26	..	1.9	1.7	1.29	6,700	2500	550
un-metal	16	..	..	2.8	..	1.8	4,870	1650	546
ine	5.8	2.9	.3	..	..	..	625	40	36
ak	6.7	4.4	1	..	..	..	670	36	48
eather	1.87	..	..	..	..	..	11	..	..

* This value is for tension, for compression E is 5800.

NOTE.—Very various estimates may be made of the elastic strength of a material, ording to the definition adopted of the elastic limit, and according to the character of load. On this question advanced students are referred to chap. xix. of the larger atise, including the Additional Notes given in the second edition (1890). The ts are fully detailed in Professor Unwin's excellent treatise on the *Testing of the aterials of Construction.* The results given above are chiefly taken from tables by ofessor Unwin.

CHAPTER XXII

FRAMEWORK STRUCTURES—SIMPLE TRIANGULAR FRAMES

In chaps. xvi. and xvii. we have investigated the stresses in the bars of a certain class of structures, built up of bars jointed together, but the methods there used applied only to the special class dealt with; so we now proceed to consider a more general treatment.

The meaning of the word structure is well known and requires no definition; the only point requiring notice being, that we shall use the word to denote a construction of bars which is not intended to move, but to keep its shape under certain loads. A roof then is a structure, while a steam engine is not.

For "framework," however, we shall require a little more consideration. The general meaning of framework is a structure built up of bars connected together in any way; but the meaning we shall in the present work attach to it is very much more limited.

Looking back to chap. xvi. we find that at the commencement of the work certain suppositions were made as to the nature of the connections of the various bars, these being—1st, the bars are supposed to be connected together at their ends by frictionless pin joints; 2d, no bar is to be continuous through a joint.

Now it is to structures which satisfy the preceding conditions that we shall confine the meaning of the word framework; actual structures, which, in most cases, fail

to satisfy these conditions, being known as girders, trusses, etc.

We now proceed to show why the preceding limitations are necessary; and we will assume in the first instance, in all cases, that the loads are applied directly to the joints of the structure by bearing on the pins which connect the bars together.

Direction of Stresses on Bars.—From what we have just been saying, it follows that the only forces which act on any bar of a framework are those applied to it by the pins which pass through its ends.

For if AB (Fig. 324) be any bar, there is no load applied to it between A and B, nor does it pass through any joint by the second assumption. From this and the first assumption follows a very important proposition, viz. *The force exerted by any bar of a framework, on the pin through its end, is in the direction of the axis of the bar.* This we proceed to prove.

Let AB be any bar of a loaded structure; then each pin exerts some force or forces on the bar, and these forces together keep the bar in equilibrium. Now the resultant action of the pin B must be a single force, say P, passing through its centre; because, there being no friction, the forces exerted by the pin on the sides of the hole are all normal (Fig. 325), thus they all pass through O_B, the centre of B, and therefore so also does their resultant.

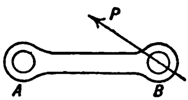


Fig. 324.

Fig. 325.

P then must be balanced by the action on AB of the pin at A. But as we have drawn it this is impossible, because one effect of P is to tend to turn AB round A, and to this the pin, being frictionless, can offer no resistance. It follows then that P can only be balanced by the action of A when its direction passes through O_A (Fig. 326), and in that case the action of A must be an exactly equal and opposite force P.

Putting it slightly differently we may say that the actions of A and B, being each single forces through O_A and O_B respectively, can balance in two ways only; they must be equal and opposite and in the same line, *i.e.* O_A, O_B, and either both outward as Fig. 326 (*a*), or both inward as (*b*).

The forces we have just drawn are those exerted *by the pins on the rod;* and it follows that the forces exerted *by the rod on the pins* must be as Fig. 327 (*a*) and (*b*) in the two cases respectively.

The line O_A, O_B we call the axis of the bar. If the bar be straight then O_A, O_B will be its axis; but the truth of the foregoing work will not be affected if the bar be—as it is sometimes in practice—curved; in that

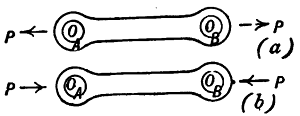


Fig. 326.

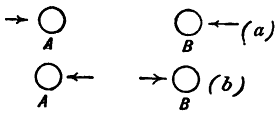


Fig. 327.

case, however, by the axis we must be understood to mean O_A O_B, and not the geometrical axis.

We shall now be able, by use of the principle just proved and of the principles of Statics, to determine the stresses in the bars of very large classes of structures. Our results will be practically exact in all cases in which the joints are actually pin joints, the only deviation from perfect accuracy being due to the friction of the joints, which should be small. If, however, the joints be riveted, or bars continuous through joints, then our results, although they may differ considerably from the truth, yet furnish us with a first approximation to the correct results; and we then proceed to consider how they must be corrected. This latter we shall, however, not be able to enter into.

Before proceeding to consider actual examples, it

will be well to consider why the preliminary work has been necessary. Let A be the centre of a pin and W the load on the joint (Fig. 328). Now there cannot be less than two bars meeting at the joint; let these be CA and BA, representing them simply by their axes.

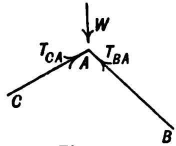


Fig. 328.

We have now the pin at A kept in equilibrium by three forces, viz. W, the action of BA, the action of CA. Now from the preliminary work we know that these latter are forces in the directions of BA and CA respectively, and hence we can find their magnitudes. But without our suppositions we should have known neither the directions of the forces nor their magnitudes; hence we should have had four unknown quantities, and only two equations—got by resolving in any two directions—connecting them, so that we could not have determined them.

We will now commence our examination of actual examples with

The Simple Triangular Frame.—ACB is a simple

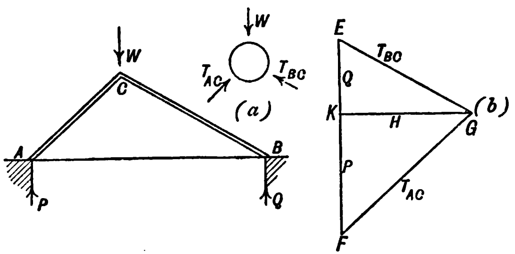


Fig. 329.

triangle of bars, which may, for example, be used to support a small roof (Fig. 329).

Let this be loaded at C and supported at A and B.

[Loads at A or B need not be considered, since they would bear directly on the supports, causing no stress in the bars.]

The principle which we shall almost exclusively proceed on will be to consider the equilibrium of the pin at one or more joints of the frame.

We will then commence with the pin at C. This is in equilibrium under the load W and the actions of AC, BC, these being in the directions of AC, BC respectively, and denoted by T_{AC}, T_{BC}. For clearness we have in (*a*) drawn the pin separately with the forces on it. Evidently they are both thrusts.

Graphic Method—Diagram of Forces.—We shall now find the values of T_{AC}, T_{BC} by graphic construction, based on the application of the proposition of Statics known as the Triangle of Forces. C the pin is kept in equilibrium by three forces, hence if a triangle be drawn with its sides respectively parallel to these forces, the lengths of the sides will be proportional to the forces.

In (*b*) we have drawn EFG such a triangle, and hence we have

$$EF : FG : GE = W : T_{AC} : T_{BC}.$$

But the triangle, as we have drawn it, represents rather more than this.

For evidently any number of triangles can be drawn having their sides parallel to the three forces, differing in size but not in proportion. Let us then draw our triangle of such a size that EF is not only proportional to W, but actually on some convenient scale represents W—this we can do by choosing our scale and then drawing the side EF first, of the proper length—then on this same scale it follows that FG and GE represent T_{AC} and T_{BC} respectively.

We now then state our method thus: Choose a convenient scale of force or weight. Draw EF parallel to W, and of such a length as to represent W on the

scale selected. From E and F draw EG, FG parallel to T_{BC}, T_{AB} respectively.

With the given scale measure EG, FG, and their values give T_{BC}, T_{AB} respectively.

For example, in the figure (Fig. 329), suppose W be 5 cwt.; EF as there drawn is 1 inch long, so the force scale is

$$1 \text{ inch} = 5 \text{ cwt.}$$

If now EG be measured it will be found to be $\frac{7}{9}$ inches,

$$\therefore\ T_{BC} = \tfrac{7}{9} \times 5 \text{ cwts.,}$$
$$= 3.9 \text{ cwts.}$$

And FG is $\frac{11}{12}$ inches, whence

$$T_{AB} = \tfrac{11}{12} \times 5 = 4.6 \text{ cwts.}$$

We have now one bar left, viz. AB. To determine the stress in this we must consider one of its end joints. We will take A. Then at A we have again three forces, viz. T_{AC}, P the supporting force, and T_{AB}. Of these we know one, viz. T_{AC}; and so by drawing a line to represent T_{AC}, and then completing the triangle of forces for the joint, or pin, at A, we can determine the other two. But we have already a line drawn representing A_{AC}, viz. FG; we will then utilise this line, and complete the triangle for A by drawing GK parallel to AC, meeting FE, which is already drawn parallel to P—*i.e.* vertical —in K. FGK is now the triangle of forces for A on the same scale as EF was originally drawn to represent W. Hence $H_{AC} = GK$ on the force scale.

[We use H here, the bar being horizontal.]

Also

$$P = FK.$$

But

$$P + Q = W = EF,$$
$$\therefore\ Q = KE.$$

And we have now determined, purely by measurement

without any necessity for calculation, the stresses in all the bars of the frame and the magnitude of the supporting forces.

The figure (*b*) as a whole is called the Diagram of Forces for the frame, and it will on examination be found to contain a triangle of forces for each joint. Two joints we have considered, the third B not being examined, since we had obtained all possible results without; if we look, however, at the triangle EKG, we see at once that it is the triangle of forces for B, its sides being parallel to Q, H_{AB}, and T_{BC} respectively, and, as we have already seen, represent them on the scale of force.

The graphic method of calculation, of which we have just had an example, is of great value, and is now very much used. In some cases, such as we have just considered, the result can be easily obtained by pure graphic work; but in some cases a little preliminary calculation, such as finding P and Q in the ordinary way by taking moments about B and A (Fig. 329), very much shortens the work; also in many cases the use of a purely graphic process tends to make the student lose sight of the principles underlying the subject, the process becoming almost purely mechanical. We shall then leave the study of pure graphic work to works dealing specially with the subject, which can be perused with most profit after the student has thoroughly mastered the principles which govern equally all methods.

With regard to the simple cases we shall deal with, one or two points must be insisted on.

1st. The student should always commence by drawing accurately to scale the structure itself, and draw on it the directions of the load or loads on it. Then the force figure should be drawn with its lines accurately parallel to the corresponding lines in the structure; and it should be drawn, not as a whole, but triangle by triangle; bearing clearly in mind as each triangle is drawn what joint it belongs to. In no case should the force figure be

drawn by use of the given dimensions of the frame without first drawing the frame itself.

For example, suppose the slopes are given as 30° and 45°.

Then it is a common practice with beginners to take (Fig. 330) EF equal to W, and draw GEF = 60°, GFE = 45°; and so obtain the figure.

This gives the correct figure, and will save perhaps a few seconds; but in cases only a little more complex it would be quite impossible to proceed in this way; and hence the student should accustom himself from the commencement to the use of correct methods for even the simplest cases, and they will then come naturally to him when dealing with more complex questions.

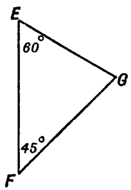


Fig 330.

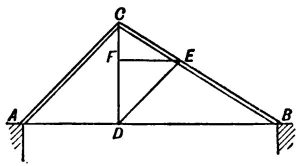


Fig. 331.

2nd. Always keep the force diagram quite separate from the figure of the frame. For example, to economise time, CD (Fig. 331) is utilised to represent W, and CDEF drawn as the force diagram. This should never be done—1st, for similar reasons to those just stated; and 2nd, because it leads to a confusion of scales. CD is so many feet long and is drawn on a linear scale; if now we take CD to represent W, then, in the first place, the scale of W will not be a scale at all in the proper meaning of the word; *e.g.* if CD, say $1\frac{1}{16}$ inch long, represent W, say 30 cwt., then the so-called force scale will be

$$\tfrac{17}{16} \text{ in.} = 30 \text{ cwt.},$$

or

$$1 \text{ in.} = \frac{16 \times 30}{17} = 28.23 \text{ cwt.},$$

which is not a scale at all; by a scale we mean such as 1 in. = 1 cwt. or 5 cwt. or 10, 12, 20, etc., but not odd numbers and decimals. Next, there will always be a likelihood of measuring some line on the wrong scale, confusing feet with pounds, and *vice versâ*.

It is instructive to examine the stresses in the triangular frame by the ordinary methods of Statics, viz. resolving the forces on the pin instead of drawing a triangle. This we will leave to the student, recommending him to solve the questions at the end of the chapter by both methods, and he will then see their relative advantages. One result, however, can be put in a simple form for calculation, by examining the diagram of forces, viz. H_{AB}, called sometimes the thrust of the frame.

Looking at Fig. 329, let $AD = a$, $BD = b$, and CD the height $= h$. Then

$$\frac{H}{K} = \frac{GK}{EK} = \frac{BD}{CD},$$

$$\therefore \; H = Q \cdot \frac{b}{h}.$$

But, taking moments about A,

$$Q(a+b) = Wa,$$

$$\therefore \; Q = \frac{Wa}{a+b},$$

and

$$H = \frac{Wab}{(a+b)h}.$$

A formula which is often useful.

We will now consider some practical examples of the use of the triangular frame.

Triangular Roof Truss.—In this case the slopes, *i.e.* the angles CAB, CBA, are usually equal. AC, BC are the rafters, and AB a tie-bar (Fig. 332).

The load is due to the weight of the roofing material which rests on the rafters. Let then

x = distance apart of consecutive rafters along the roof in feet,
w = weight per sq. ft. of roofing in lbs.

Then the load on AC is

$$wx \,.\, \text{AC lbs.}$$

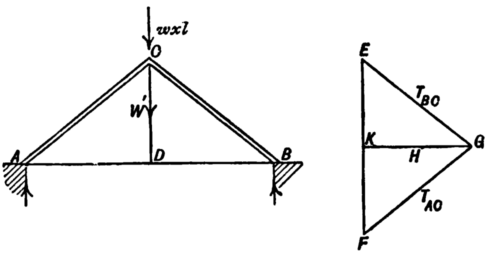
And this rests half on C and half on A. Similarly the load on BC is

$$wx \,.\, \text{BC lbs.},$$

resting half at C, half at B. Hence

$$\text{Total load at C} = wx\frac{\text{AC}+\text{BC}}{2},$$

which applies generally, whatever be the slopes.



(a) Fig. 332. (b)

Thus for equal rafters each l ft. long, the load at C is wxl (Fig. 332).

We have here drawn an extra bar, viz. CD; such a bar is usually fitted, helping to take the weight off the ceiling, which hangs to AB; the portion of such weight which is carried by CD then becomes an additional load on C, to which CD hangs, and must be added to the other load, and the diagram of forces then drawn in the usual manner.

If W′ be the load on the bar CD, we have

$$\text{Total load at C} = W' + wxl,$$

and (*b*) then represents the diagram of forces, EF being made equal to $W' + wxl$.

Thrust of the Roof.—In some cases the tie-bar AB is omitted, and then the resolved parts of T_{AB}, T_{BC} must be balanced by inward thrusts of the walls, these thrusts being equal to GK in the diagram of forces, *i.e.* to what H_{AB} would be if there were a tie-bar. The roof then thrusts the walls outward with forces equal and opposite to H_{AB}, the magnitude of each being called the Thrust of the Roof; and the walls must be made strong enough to withstand this thrust.

In the particular case of equal slopes the expression for the thrust is simple. For we have, on page 444, the formula

$$H = \frac{Wab}{(a+b)h}.$$

If now l = span of roof,

$$a = b = \frac{l}{2}.$$

Whence

$$H = \frac{W\frac{l^2}{4}}{lh} = \frac{Wl}{4h}.$$

Bridge Truss.—Fig. 333 represents the preceding type of truss inverted and used to support a bridge platform. The inversion of the truss does not affect the shape of the force diagram, *i.e.* the magnitudes of the stresses in the bars, but it inverts their natures; thus AC, BC are now ties, and AB a strut. Also the bar CD is now a necessary part of the structure, since the load rests primarily on the platform, and is transmitted to the joint C by means of the strut CD.

The platform AB acts as a strut, and is not in practice jointed at D, but for our purpose we must assume such a joint or our method of calculation fails. This assumption credits AB with no power in itself of supporting a load, and hence errs on the safe side so far as the values of the longitudinal stresses in the bars are concerned.

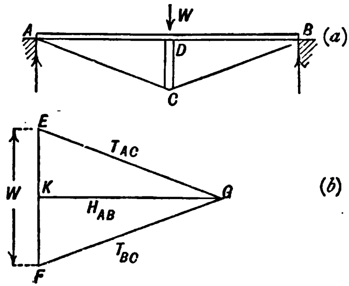


Fig. 333.

1st Case.—Let a single load W rest at D.

Then considering the joint D,

$$T_{CD}=W.$$

[Actually the stiffness of AB would make T_{CD} less than W.]

But T_{CD} is the load at C, hence C is loaded with W, and (*b*) (Fig. 333) gives the diagram of forces.

2d Case.—Uniformly distributed load, w lbs. per foot run, on platform. Then

$$\text{Load at D}=w\frac{AD+BD}{2}=\frac{wl}{2},$$

$$\therefore\ T_{CD}=\frac{wl}{2}.$$

And the diagram (*b*) is drawn for this load at C.

The supporting forces as found from the diagram are each $wl/4$ instead of $wl/2$, which we should naturally call the supporting forces for a total load wl; the discrepancy is due to the fact that there is a load $wl/4$ at each end, in addition to $wl/2$ at D (see page 323 for a full explanation).

If the strut CD be short, then it will be found difficult

to determine the correct position of the intersection of EG and FG, and in such cases the calculation method is preferable.

Derricks and Cranes.—Fig. 334 shows the triangular frame used as a derrick.

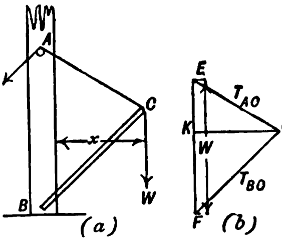


Fig. 334.

AB is a portion of a mast; BC is the *jib*, jointed at B to the mast; and AC is a rope which can be used to raise or lower the jib, and is called the *Topping Lift*. The load hangs at C.

Fig. 335 shows an ordinary portable crane. There is now no mast, so a *Crane Post* AB must be provided, and this being short

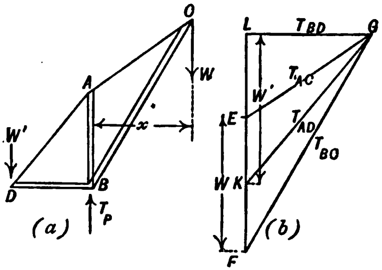


Fig. 335.

compared with the height of a mast, AC slopes upward, and is now called the Stay.

The diagram of forces for the joint C is shown in the two cases, and we see that the effect of sloping AC upward is to increase both T_{AC} and T_{BC}.

Let us now leave the joint C, and examine the actions on the joint A. We have T_{AC} acting on A, which may

be resolved into a vertical and a horizontal component; the former of these produces a longitudinal stress in AB, and the latter tends to overturn the mast or crane post by turning it round B. The overturning effect can also be seen by considering the triangle ABC; for there is a moment $W \times x$ in each case tending to rotate the triangle as a whole round B. There are various methods of preventing the overturning, and in this lies the chief difference between various types of crane.

In the case of the derrick the mast is securely fastened to the deck, and by the stays, and so does not overturn.

In Fig. 335 overturning of the post is prevented by the backstay AD, which is attached to A, and to a point D in BD, the platform of the crane. This would still leave the crane liable to turn over as a whole, although the post could not turn relatively to the platform; and so to prevent this it is necessary to load the platform at D with a weight W′.

We will now proceed to determine by a diagram the stresses in backstay, post and platform, and the magnitude of the balance weight required.

Considering the joint A, the forces are

$$T_{AC}, \quad T_{AB}, \quad T_{AD}.$$

And T_{AC} is already represented by EG.

Draw then EK parallel to AB, and GK parallel to AD. Then

$$T_{AB} = KE, \quad T_{AD} = GK.$$

We could not now see, *a priori*, whether T_{AB} would be a thrust or pull, but we can determine this from the figure. For the sides of the triangle of forces represent the directions of the forces which keep a body in equilibrium *if we go round the triangle in the same direction*, *i.e.* forces represented by

EG, GK, KE,

or by

GE, EK, KG,

would be in equilibrium, but not those represented by say

GE, KE, GK,

since here we do not go round in one direction only.

Now, to apply this, we note that the T_{AC} which acts on the joint A is represented by EG—not by GE, GE representing the T_{AC} which acts on C—we must go round then in the first way above, *i.e.*

EG, GK, KE,

which shows that T_{AB} or KE is an upward force on the joint A, which we are considering, so that it is a thrust. Also T_{AD} or GK acts to the left and is a pull, but this we could see at first, whereas the direction of T_{AB} is not so easily seen, depending on the slopes of AC and AD, and it may in some instances be a pull.

We will now proceed to the joint at D. Forces

$$T_{AD},\quad W',\quad T_{BD}.$$

And T_{AD} on D is represented by KG, so we draw GL parallel to BD, and KL parallel to W′. Then

$$T_{BD}=GL,\quad W'=LK.$$

We have now found all the stresses, and considered the equilibrium of all the joints except B. If we examine our diagram, however, we shall find that a triangle showing the equilibrium of B already appears in it.

For the forces acting at B are T_{BD}, T_{AB}, T_{CB}, and the upward thrust T_p, say of the pivot at B, on which the crane turns.

Of these T_{AB} and T_p are in one line, so their resultant is $T_p - T_{AB}$ upward, and taking this as one force we now have only three forces.

But we have already a triangle, viz. LFG, with sides parallel to these three, and of which two sides, LG, GF, have already been seen to represent T_{BD} and T_{BC}; hence the third side FL must represent $T_p - T_{AB}$.

From this we have

$$
\begin{aligned}
T_p - T_{AB} &= FL, \\
\therefore\ T_p &= T_{AB} + FL, \\
&= KE + FL, \\
&= KL + FE, \\
&= W' + W,
\end{aligned}
$$

and this must evidently be correct, since the pivot supports the whole weight resting on the crane, *i.e.* W + W'.

In practice, a large part, or the whole, of the counterbalance is provided by the weight of the boiler and engine used for driving the lifting winch.

Fixed Cranes.—In the crane just described, the jib, stay, and backstay are always in one plane, since the platform turns carrying the whole with it; there is then no tendency to turn the crane over sideways when the load hangs quietly; and although when the crane is turning there is a tendency to pull it over sideways, it is but small, and is provided against by the stiffness of the pivot and the resistance of its bearings.

If, however, the crane post be simply pivoted in a bearing fixed to the ground, then if a single backstay only were provided, this could only balance the overturning effect of the weight when it was in the same plane as the jib and stay, as shown in plan by Fig. 336 (*b*), and then, when the jib was turned to a new position, as (*c*), there would be an overturning moment Wx, which the backstay could not prevent from overturning the crane.

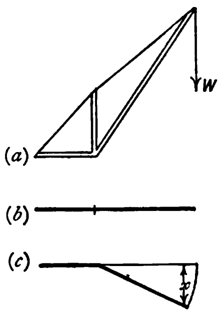


Fig. 336.

To overcome this difficulty it is usual to have two backstays, as Fig. 337, AE, AD being the stays, always equal, and fastened to the ground at D

and E. The combined action of the two then renders overturning impossible in any direction.

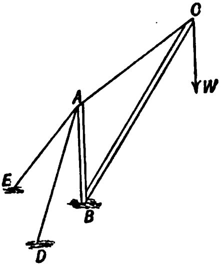


Fig. 337.

We wish now to find the stresses in these stays, and we have a different case from any before dealt with, because the bars are not all in, or parallel to, one plane.

We shall require to draw more than one view of the structure, and we will first consider the plan (Fig. 338). A is the crane post, E and D the points to which the stays are fastened, AE, AD a plan of the stays, AC the given position of the jib.

Now to prevent any overturning of the post, what is necessary is that the resultant action of the two backstays should be in the same plane as the jib and stay, *i.e.* if we produce CA to F, then the plan of the resultant of T_{AE} and T_{AD} must lie along AF. There will then be no force causing overturning at right angles to FAC.

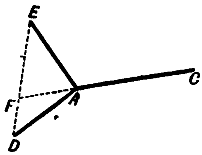


Fig. 338.

We may express this by saying that the two backstays must be equivalent to a single backstay, which lies along AF in plan.

But now the resultant of T_{AD}, T_{AE} must be in the plane of AD and AE, so that it must be the line joining the top A of the crane post to the point F on the ground, this being the only line which is at once in the plane of jib and stay, and also in the plane of the two backstays. We say then that AD and AE are equivalent to the single backstay AF.

If then we replace AD and AE by AF, and then find the stress R in such a backstay, this stress R represents

the resultant of T_{AD}, T_{AE}, and we can then find its resolved parts, which will give us the required results.

We proceed then as follows: Draw a correct plan of the crane in the given position (Fig. 338), join DE and produce CA back to cut DE in F.

To determine R, draw a side view of the crane and the single equivalent backstay (Fig. 339).

To construct this figure:

Draw BA, the crane post; AC, the stay; and BC, the jib.

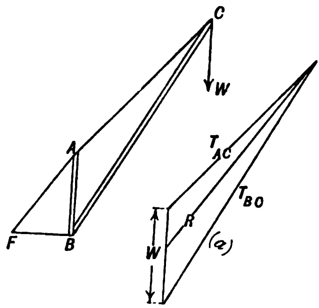


Fig. 339.

Next, from Fig. 338, take the length AF and set off BF equal to it at right angles to AB, *i.e.* along the ground.

Finally, join AF.

Fig. 339 is a correct side elevation of the crane and equivalent backstay.

Draw the diagram of forces (*a*), giving the value of R.

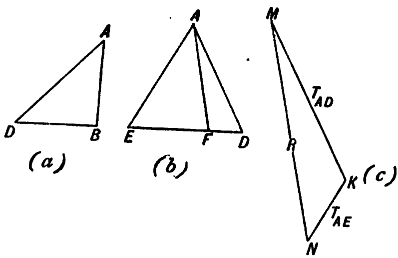


Fig. 340.

We have now to resolve R along AD and AE, for which purpose we require a correct view of the backstays in their own plane (Fig. 340).

For this construction draw in (*a*) AB the crane post, and make BD equal to AD in Fig. 338. Then joining AD we have the correct length

of each backstay. We take then DE from Fig. 338 in (*b*), and make DA, EA each equal to AD in (*a*). Also mark from Fig. 338 the position of F and join AF.

Then Fig. 340 (*b*) is a correct view of the backstays and the equivalent single backstay in their own plane.

We now then draw in (*c*) MN equal to R from Fig. 339, and drawing MK, NK parallel to AD, AE respectively, we have finally

$$T_{AD}=MK, \quad T_{AE}=KN,$$

which are the required results.

The point F (Fig. 338) may come anywhere in the line DE, and as AC revolves it will fall outside the backstays. In this case it can be easily seen that one of the stresses T_{AD}, T_{AE} must be compressive, and hence the stays must not be chains, but solid bars or spars. This is actually the case in practice.

Sheer Legs.—In this case (Fig. 341) there is no post, but the stay is prolonged to meet the ground, thus having stay and backstay in one. A nut is formed at the end A, which works on a screw, the rotation of the screw running A in or out and so raising or lowering C.

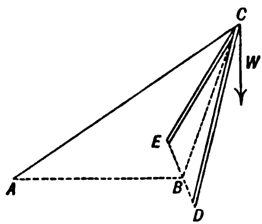


Fig. 341.

Overturning sideways is now prevented by replacing a single jib as CB by two legs, DC, EC, pivoted at D and E respectively.

To find the stresses we proceed in an exactly similar manner to the preceding. The actual legs are imagined replaced by the single leg BC, which is now always midway between them, since the vertical plane through AC and W bisects DCE.

We then find the stress for AC and BC by drawing the triangle of forces for C. And, finally, resolve T_{BC}

into its two components, T_{CD}, T_{CE} exactly as in the previous case.

By considering the joint A we can determine the thrust along AB, the axis of the screw and the upward pull on its bearings being respectively the horizontal and vertical components of T_{AC}.

Tension of Lifting Chain.—We have hitherto considered the load W as simply hanging to C, but this is not the case in practice, W being always supported by a tackle, the rope or chain of which passes over a pulley at C and hence to a chain barrel.

In Fig. 342 we represent such a case, CL being the chain.

Let now the force ratio of the tackle be n (page 115).

Then

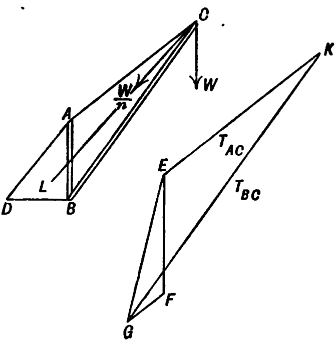


Fig. 342.

$$\text{Tension of chain} = \frac{W}{n}.$$

Now C is loaded with two loads, W and W/n, and we must first find the resultant load at C.

We draw then in (a)

$$EF = W, \quad FG = \frac{W}{n},$$

these being parallel to W and CD respectively. Then joining EG, EG is the resultant load at the joint; and we now draw EK, GK parallel to AC, BC respectively, these giving T_{AC}, T_{BC}, and proceed with the diagram of forces as before.

In many cases the chain runs on pulleys on the stay, so that its direction is that of the stay. In this case the ordinary diagram for W alone is drawn (Fig. 335), and

then T_{AC} is composed of the real tension in AC and the tension of the chain which assists it. Hence

$$\text{Tension in AC} = T_{AC} \text{ (from diagram)} - \frac{W}{n}.$$

If the chain were led down the jib, its tension would increase the thrust on the jib, and we should have

$$\text{Thrust in BC} = T_{BC} \text{ (from diagram)} + \frac{W}{n}.$$

Bending of Crane Posts.—In small cranes backstays are often omitted, the crane post being made stiff and supported in bearings. Fig. 343 shows the post prolonged, B and D being the bearings.

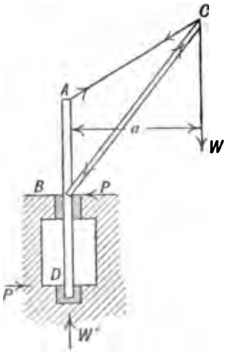


Fig. 343.

Each bearing then exerts a horizontal thrust P, and the moment P. BD balances Wx, the overturning moment,

$$\therefore \text{P.BD} = \text{W}a,$$

and

$$\text{P} = \frac{\text{W}a}{\text{BD}}.$$

In addition D supports the weight of the crane and load.

The preceding considers the crane as a whole; but now consider the post alone. Then the overturning forces are the horizontal components of T_{AC}, T_{BC}, these being equal (Fig. 335), and each equal to Q say.

Then (Fig. 344) we have two couples, Q . AB, P . BD, keeping the post in equilibrium. So that

$$\text{Q.AB} = \text{P.BD} = \text{W}a.$$

The post is now subjected to bending and shearing, due to a load P + Q at B, supported by P at D and Q at A. Whence

$$M_B = \frac{(\text{P}+\text{Q})\text{AB.BD}}{\text{AD}},$$

and the figures (*a*) and (*b*) show the curves of B. M. and S. F.

Fig. 345 shows another method of supporting the post, there being a bearing at each end. In this case there is practically no bending or shearing.

There is a very large variety of types of crane, but space will not permit of our examining any more

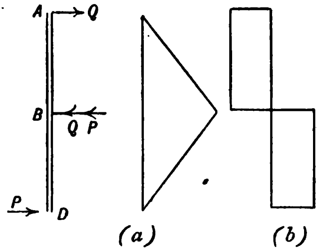


Fig. 344.

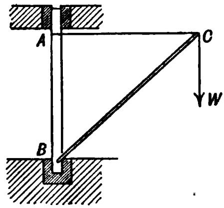


Fig. 345.

examples; in all cases the methods used in the present chapter will be found to apply.

EXAMPLES.

1. A simple triangular truss, 24 feet span, 3 feet deep, is supported at the ends, and carries a load of 3 tons concentrated in the middle. Find the stress on each member.

Ans. 6.2; 6 tons.

2. The span of a roof is 15 ft., length of rafters 9 ft. and 12 ft. respectively. The rafters are 2 ft. apart along the roof, and the roofing material weighs 15 lbs. per sq. ft. Find by construction the thrust of each rafter and the stress in the tie bar.

Ans. 189; 252; 151 lbs.

3. A pole 40 ft. long is used as a derrick. One end rests on the ground, the other is supported at a height of 30 ft. from the ground by a chain at right angles to the pole. If the chain will safely bear a pull of 3 tons, what weight can be lifted by the derrick?

Ans. $4\frac{1}{2}$ tons.

4. The jib of a derrick is inclined at 45° to the vertical, and

the topping lift is attached to a point vertically over the foot of the jib at a height equal to its length. Find by construction the pull on the topping lift and thrust of the jib when lifting 4 tons.

Ans. 3.1 ; 4 tons.

5. The horizontal member of a simple triangular truss 10 ft. span, 3 feet deep, is loaded with 2 tons, uniformly distributed over half the span, and is supported at the ends. Find the stresses in the bars. *Ans.* 1 ; .83 ; .97 tons.

6. A crane has a vertical crane post AB 8 ft. long, and a horizontal tie BC 6 ft. long, AC being the jib ; it turns in bearings at A and B, and the chain supporting the load passes over pulleys at C and A, passing from A to the chain barrel at an angle of 30° to AB. Find the stresses in the bars and thrusts on the bearings when raising 1 ton at a uniform rate.

Ans. Thrust on jib = $1\frac{1}{4}$ ton.

7. The crane post AC of a crane is 8 ft. high, AB the jib is 21 feet long, and there are two tie bars CB each 16 ft. long. The load of 5 tons is supported by a pair of two-sheaved blocks, the chain passing to the chain barrel bisecting AC. The backstay is inclined at 45° to AC. Find the stresses in all the bars, and the magnitudes of the balance weight at the foot of the backstay.

Ans. Jib, $13\frac{3}{7}$; stay, $9\frac{3}{8}$; post, 4 tons ; backstay, 11.8 ; counter-balance, $8\frac{3}{8}$.

8. In a crane, the jib is twice as long, and the stay one and a half times as long, as the crane post. The backstay is parallel to the jib. The crane post turns in bearings at its foot, and is produced below the ground to a distance equal to its height above where it turns in the foot step bearing. The end of the platform, where the backstay meets it, is stayed to the end of the crane post at the footstep, there being no balance weight. Find all the stresses and thrusts on the bearings, when loaded with 10 tons.

9. A fixed crane post is 20 feet high, the stay is 20 ft. long, and the jib 32 feet. There are two backstays each 24 feet long, their plan being a right-angled triangle. Find the stresses when lifting 10 tons, the plane of jib being midway between the stays, and show how the stress on the backstays changes when the jib has swung through 45° and 90° from its first position.

Ans. Jib, 16 ; stay, 10 tons.

10. A pair of sheer-legs are 60 ft. high when upright, each leg being 52 ft. long. The back leg is 90 ft. long. The front legs swing round pivots 3 ft. from the dock side. Find the greatest stress in each leg when placing a boiler weighing 20 tons on board a vessel of 40 ft. beam, the hatch being in the middle line of the ship. *Ans.* Back leg, 16 ; front leg, $16\frac{3}{4}$ tons.

PART III

HYDRAULICS

CHAPTER XXIII

HYDRAULICS

WHEN a force is applied to a solid body a certain definite change of shape (omitting cases of actual breakage or deformation) is first produced, and the body then moves as a rigid body, the particles of which it is composed maintaining their relative positions unchanged. Finally, on removing the force, the body returns to its original shape.

In the case of liquids, which we are now about to consider, we have the exact opposite to the above; there is no definite amount of change of shape produced when a given force is applied, but the amount is anything we please, depending entirely on how long the force is applied. Also the small particles do not keep their relative positions but move freely about among one another.

If then we attempt to determine the motion of a fluid or liquid by simple application of the principle of work and the laws of motion, we should have to inquire into the motions of these small particles. This is done to a certain extent in the branch of mechanics called **Hydrodynamics.**

The results of hydrodynamics can be rarely made available for practical purposes, and hence we must call actual experiment to our aid. We then treat, not of the motion of each particle, but of the water or other liquid as a whole, the results sought for being obtained by a

combination of experimental facts with theoretical reasoning. The science which treats the subject in this way we term **Hydraulics**. We shall, with one exception, confine ourselves exclusively to one liquid, viz. water, though the methods will be applicable to all.

Effect of Gravity—Head.—We commence by considering the effect of gravity on water, since a large part of the flow of water takes place under gravity alone. In Fig. 346 ABCD is a reservoir, AB being the water level.

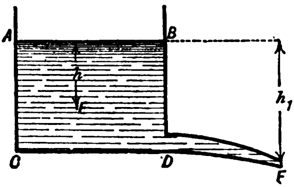


Fig. 346.

Take now any point E in the reservoir h feet below the water level, then we know from hydrostatics that, due to the weight of the water above E, there is at E a pressure the magnitude of which on every square foot is equal to the weight of a column of water 1 sq. ft. in section h ft. high. If, then,

P = pressure per sq. feet at E,
w = weight of 1 c. ft. of water in lbs.,
$$P = wh \text{ lbs.}$$

The fact that E is h ft. below AB we express in hydraulics by saying there is a **Head** of h feet at, or over, E, and this is, we see in the case of still water, equivalent to saying there is a pressure of P or wh lbs. per sq. foot at E. The pressure is plainly present on a horizontal plane at E, and it can be shown both experimentally and theoretically that it is also present as a direct normal pressure on a plane passing through E in any direction; if the plane be not horizontal the intensity of the pressure varies from point to point, but *at* E is wh lbs. per sq. foot.

[Notice that we use now lbs. per sq. ft., not per sq. inch.]

We have taken E in the body of the reservoir, but this was not necessary. For let there be a long pipe as shown (Fig. 346), then at F there is a head h_1, and a pressure produced by it wh_1; and similarly for any other point so long as there is a continuous fluid connection, *and the water is at rest.* The latter qualification is, as we shall see, of the first importance.

The pressure produced by a given head depends on the density of the liquid ; in the case of water we take

1 c. ft. of fresh water weighs 62.5 lbs.
1 c. ft. of sea water weighs 64 lbs.

Thus a head of

$$1 \text{ ft. of fresh water} = 62.5 \text{ lbs. per sq. ft.,}$$
$$= \frac{1}{2.3} \text{ lbs. per sq. inch.}$$
$$1 \text{ ft. of sea water} = \frac{1}{2.25} \text{ lbs. per sq. inch.}$$

In some cases mercury is used to balance or measure a pressure, the head being then generally measured in inches, and then

$$1 \text{ inch of mercury} = .49 \text{ lbs. per sq. inch.}$$

We have in speaking of the pressure at E omitted the atmospheric pressure, so that the real pressure at E is wh plus that of the atmosphere.

The mean value of the atmospheric pressure may be taken as 14.7 lbs. per sq. inch, or 2136 lbs. per sq. ft., and it is thus equivalent to a head of 34 feet of fresh water, or 33 feet of salt water.

We do not, however, include the atmospheric pressure when we speak of the head at E ; by that term we shall always be understood to mean the depth below the water surface.

Unresisted Flow under a given Head.—ABCD is now a reservoir, from which water is flowing through

the open end of the pipe DE (Fig. 347), and we suppose either that the reservoir is so large that the water level remains practically constant, or that it is maintained constant by water entering from some other source. Let

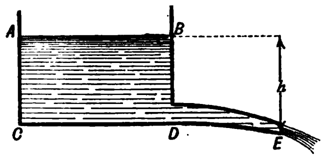


Fig. 347.

$$h = \text{depth of E below AB in feet,}$$

then the head over E is h feet.

The question we wish to answer is: Suppose there be no resistance offered to the motion, with what velocity will the water flow out at E?

[We see now why it was necessary on page 463 to insist on the water being still, for there is now no pressure produced by the head, since the stream leaving at E is exposed to the atmosphere (see page 465), the head being now a source of velocity.]

Let us consider the time during which 1 lb. of water, which was all originally at the water level, falls to E and out at E. Then, there being no source of effort but gravity,

$$\text{Energy exerted} = h \text{ ft.-lbs.,}$$

1 lb. having been exerted through h ft.

No work has been done, and consequently if v be the velocity at E,

$$h = 0 + \frac{v^2}{2g},$$

$$\therefore\ v^2 = 2gh, \quad \text{or } v = \sqrt{2gh}.$$

Velocity of Flow.—The question now arises—What velocity does v really represent? To understand the point of this question, let us proceed to consider how much water flows out at E per second, or to find the discharge. Let

A = sectional area at E in sq. ft.
Q = discharge in c. ft. per second.

We should then naturally put

$$Q = Av = A\sqrt{2gh}.$$

It is here the question arises. For it does not necessarily follow that, because all the particles of water leaving E have the velocity $\sqrt{2gh}$, therefore the discharge is $A\sqrt{2gh}$. The quantity of water flowing out depends on the velocity of the particles square across the end section; if the water is all flowing square across, as (*a*) Fig. 348, then $Q = Av$, but if some are flowing in one direction and some in another, as (*b*), then the mean velocity of the whole stream across the section is not v, but some less velocity V. (See note on page 494). This velocity V we call the Velocity of Flow, and we have always

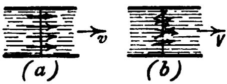


Fig. 348.

$$Q = AV.$$

The determination of the relation between V and v is one of those questions which can only be answered by direct experiment; before dealing with this, however, we have one other point to consider.

Head due to Difference of Pressure.—We see now why in speaking of head it is not generally necessary to consider the atmospheric pressure, because in the case we have just considered the atmospheric pressure acts equally as an effort and a resistance, producing, on the whole, no effect. And our previous work will hold good quite irrespective of the magnitude of the pressure of the surrounding atmosphere; thus the vessel ABCD and the pipe might be in a closed chamber exposed to a pressure above the atmosphere or to a partial vacuum, the discharge taking place into the same chamber. In some cases, however, the pressure at exit is different from that on the water surface, and we will now see what effect this will produce.

Fig. 349 (*a*) shows a reservoir in which the pressure on AB is P_1 lbs. more than that at E, either due to excess of air pressure or to a loaded piston pressing on the surface AB. In (*b*) we have the reservoir shown filled up to a height P_1/w feet above AB. Now, comparing these two it is quite impossible to conceive that there can be any difference in the motion of the water under the plane AB in the two cases. In each case there is on AB a pressure P_1 plus the general pressure around the whole system. The movement of the water depends on the pressures applied, not on what applies

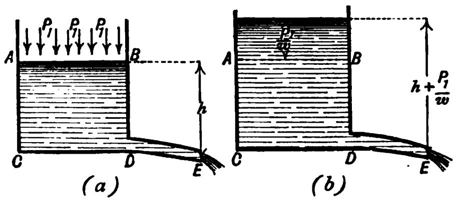


Fig. 349.

them, and will thus be exactly the same when the excess pressure is applied by water, as in (*b*), as when it is applied by air or metal, as in (*a*).

It appears then that a difference of pressure P_1 lbs. per sq. ft. is equivalent to an increase of P_1/w feet of head; and hence P_1/w is called the head equivalent to the difference of pressure at E and AB. For the flow we have

$$v^2 = 2g\left(h + \frac{P_1}{w}\right) \quad \text{(from } b\text{)}.$$

Discharge from Simple Orifice.—We now proceed to inquire in what way our preceding result requires modification in actual practice.

Fig. 350 represents a vessel discharging water through a circular hole, the inner edge of which is chamfered to a sharp edge.

First, we notice that the jet issuing contracts after passing the hole, the contraction continuing until a distance, which is roughly $d/2$, d being the diameter of the hole. The section CC, at which the contraction is complete, is called the Contracted Section, and if this be measured, which can be done by means of measuring screws fitted on each side of the orifice, its diameter will be found to be $\frac{4}{5}$ of d.

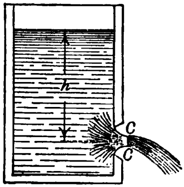


Fig. 350.

If then A_0 be the area of the orifice, and A that of the contracted section,

$$A=\left(\frac{4}{5}\right)^2 A_0 = .64 A_0.$$

This is for a circular orifice, for other shapes the ratio of A to A_0 will have different values ; we express the fact of contraction by the equation

$$A=C_c A_0,$$

C_c being called the **Coefficient of Contraction**, its value depending, as we have stated, on the shape of the orifice.

Now, for the velocity of issue, let

$$h = \text{depth of centre of hole,}$$

then if d be small compared to h, the water flows under the head h, and hence

$$v^2=2gh.$$

What we wish to determine, however, is V, the velocity of flow ; we express the fact that V differs from v by writing

$$V=C_v v,$$

C_v being the **Coefficient of Velocity**, and we require the value of C_v.

There are two ways in which this may be done,

the more simple being an indirect method. We will explain this way first. If the jet were equal in section to the orifice, and there were no cross flow, we should have the discharge given by

$$Q_o = A_o\sqrt{2gh}.$$

This we call the theoretical discharge.

Let now Q, the actual discharge, be measured, then Q will be found to be less than Q_0 in a certain ratio ; we shall have

$$Q = C_D Q_o,$$

C_0 being the **Coefficient of Discharge**, and its value for a circular orifice being .62. But

$$\begin{aligned} Q &= AV, \\ &= C_c A_o \times C_v\sqrt{2gh}, \\ &= C_c C_v Q_o. \end{aligned}$$

And therefore,

$$C_c C_v = C_D,$$

and having already determined C_c and C_0 we obtain C_v. For the circular hole

$$\begin{aligned} .64C_v &= .62, \\ \therefore\ C_v &= .97, \end{aligned}$$

and

$$V = 97\sqrt{2gh}.$$

The second method of obtaining C_v is by allowing the jet to strike a plane, and marking the position of its centre. If then (Fig. 351) CD be the jet, we have CE = Vt and ED = $gt^2/2$.

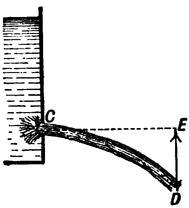


Fig. 351.

Whence, eliminating t, we obtain V in terms of CE and ED ; these distances being measured, V is then known.

Head Wasted—Coefficient of Resistance. — We have expressed the loss of what we may term *useful* velocity by a coefficient of velocity, but there is another method

of expressing it which is of more general importance. To simply produce the velocity V we should, in the absence of waste, require only a head $V^2/2g$; actually, however, the greater head h is required, hence we say the head $h - V^2/2g$ is wasted, and we write

$$h' = h - V^2/2g,$$

h' being the **Head Wasted.**

This head is wasted because the water passes an orifice, the edges of which offer a **Hydraulic Resistance,** *i.e.* a resistance to the passage of the water, and we express this fact by saying that the ratio $h' : V^2/2g$ is the **Coefficient of Resistance** of the orifice. There is a peculiarity here which is of great importance. If we denote the coefficient of resistance by F we have

$$h' = F\frac{V^2}{2g},$$

thus we have expressed the loss, not as a fraction of the original head, but as a fraction of the kinetic energy in one pound of the water. There is a reason for this which we will now give.

Law of Hydraulic Resistances.—Whenever water flows over a rough surface, or suddenly changes its velocity either in magnitude or direction, there is a hydraulic resistance to the flow, and *each of these resistances causes a waste of head which bears a fixed ratio to* $V^2/2g$, where V is the velocity of flow past the obstacle. For every kind of obstacle there is then a fixed value of F depending chiefly on the nature of the obstacle, and this is why we have expressed h' in the preceding way. As we examine other cases we shall see that this law is approximately fulfilled.

[We can see that the preceding is what we should naturally expect. For all the waste of head takes place in giving the water cross motions, and if we double V we probably increase these cross motions in nearly the same proportion, and hence quadruple the head which must be used up in producing them.]

Connection of Coefficients.—Since F and C_v both express one fact, they must be in some way connected, and the connection is easily determined. For

$$V = C_v\sqrt{2gh},$$

and

$$h' \text{ or } F\,.\,\frac{V^2}{2g} = h - \frac{V^2}{2g},$$

$$\therefore (1+F)V^2 = 2gh = \frac{V^2}{C_v^2},$$

$$\therefore F = \frac{1}{C_v^2} - 1,$$

and thus for a circular orifice

$$F = \frac{1}{.97^2} - 1 = .06,$$

so there is very little waste of head.

Discharge from a Short Pipe.—We have said that the values of the coefficients depend on the nature of the obstacles to flow; let us examine what takes place when a short pipe of length about $3d/2$ is fitted outside the hole.

In the first place, the water issues in a full stream of area A_o, and therefore C_c is unity.

In spite, however, of the gain in sectional area the discharge is found to be much less than for the simple orifice, C_D being now only .815. It follows that C_v is also .815, so there is a great falling-off in velocity.

For the resistance we have

$$F = \frac{1}{(.815)^2} - 1 = .505,$$

so that $\frac{.5}{1.5}$ or $\frac{1}{3}$ of the head is now wasted. The reason of this increased resistance is shown at page 486.

Surface Friction.—We have in the preceding case given a certain value to the head wasted, but we do not know exactly how it is wasted, whether from one cause only, or from more than one. We shall now proceed to consider certain causes which waste head separately, commencing with the most important.

Fig. 352 shows a thin plate with sharp edges completely immersed in water or other fluid, through which it is moving edgeways at V f.s. In order to keep the plate moving at V f.s. it is found that an effort R is required, this effort balancing the friction of the water on the surface of the plate.

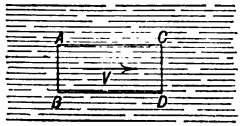


Fig. 352.

The value of R is determined by experiment, and it is found that

$$R = fwS\frac{V^2}{2g},$$

where

w = weight of 1 c. ft. of the fluid (including gaseous fluids),
S = surface of the plate,

and f is a constant depending only on the nature of the surface of the plate, and independent of the units employed. If we compare with the ordinary laws of friction, we find they are exactly opposite. For in the friction of dry surfaces R is—1st, independent of V; 2d, dependent on the pressure; 3d, independent of the surface; and each of these is exactly opposite in fluid friction.

The fact that R is independent of the pressure should be noticed, as it is a very common thing to meet statements to the effect that there is more loss by friction of a square foot of surface at the keel of a ship than there is at the water line. Experiment has conclusively shown that this is not so. The values of f determined by Froude for a board 20 feet long, 19 inches deep, are

Nature of Surface.	Varnish.	Fine Sand.	Medium Sand.	Coarse Sand.
Coefficient . . .	.00278	.0048	.00534	.00588

S is to be taken for the two sides, *i.e.* in the present case $S = 20 \times 12 \times 19 \times 2$ sq. ins. The value of f is found to be affected by the length of the board, f being greater for short boards than for long, thus in the above table the first value for a board 2 ft. long is .0041, and for one 50 ft. long .0025. The effect is due to the fact that the first portion of the board drags the water along, and thus lessens the velocity of rubbing over the succeeding portion. These laws are of the first importance in the determination of the resistance of ships, but into this we cannot enter.

Surface Friction of Pipes.—The chief use we shall make of the preceding will be to determine the waste of head caused by surface friction of a pipe.

In Fig. 353 we show a horizontal pipe of uniform diameter, through which fluid is flowing with velocity V.

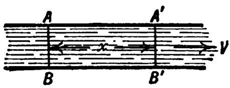


Fig. 353.

Take two sections, AB, A′B′, a distance x apart, and consider the motion of the water between them. For clearness we may imagine two pistons at AB, A′B′, which actually isolate this portion of the fluid. This portion of fluid being in uniform motion is balanced, and therefore

Effort = total resistance.

Let

p be the pressure in the pipe at AB,
p' ,, ,, ,, ,, A′B′,
A = sectional area of pipe,
s = perimeter of pipe.

Then

$$\text{Effort} = pA,$$
$$\text{Resistance} = p'A + \text{frictional resistance},$$
$$= p'A + fw(sx)\frac{V^2}{2g}.$$
$$\therefore pA = p'A + fw(sx)\frac{V^2}{2g},$$

or

$$\frac{p-p'}{w}=f.\frac{s}{A}.x\frac{V^2}{2g}.$$

There is then a loss of pressure due to the friction; but we have seen pressure is equivalent to head, and a loss of pressure $p-p'$ is equivalent to a waste of head $(p-p')/w$; hence if h' be the head wasted we have

$$h'=f.\frac{s}{A}.x\frac{V^2}{2g}.$$

The quantity A/s is called the *Hydraulic Mean Depth*, and denoting it by m,

$$h'=f.\frac{x}{m}\frac{V^2}{2g},$$

is the head wasted in a length x.

In the case of a cylindrical pipe

$$m=\frac{\frac{\pi}{4}d^2}{\pi d}=\frac{d}{4},$$

and denoting the length of the pipe now by l, we obtain

$$h'=4f.\frac{l}{d}.\frac{V^2}{2g},$$

as the total loss in the whole length of pipe.

The value of f is, as before stated, independent of the units, but looking at what we found as to the effect of length, we should be doubtful about what value to take for it. This difficulty we get over by taking the value of f, not from Froude's results, but directly from experiments on pipes; we then are sure of having the correct value. It is found that $4f$ varies from .04 for 1 inch diameter to .02 for 4 inches and upward, varying, however, considerably according to the condition of the pipe. The values above are for a clean cast-iron pipe, and as an average value we use .03. In particular cases these values may be much exceeded.

Pressure in a Pipe.—In the preceding we have used the term "pressure in the pipe," and have determined the loss of pressure. This being a question on which erroneous ideas are often expressed, it will be useful to see exactly what we mean by "the pressure in the pipe."

If a board be held in front of a jet of water issuing from a hose, a great pressure is felt on the board; or if the hand be held in a running stream a pressure is felt on it. Is it this pressure which we mean, and is it right to say, as it commonly is said, that the jet from the hose issues at high pressure?

To these questions we answer No! By the pressure in a pipe we mean the pressure of the portions of water on one another, not on a body which is held still so as to stop their flow. For example, in Fig. 353, the pressure of the water to the left of AB on that to the right is exerted on water which is moving away as fast as the pressing water follows it, and to feel this pressure we should, in the second example above, move the hand along with the stream. If we did this we know we should feel no pressure at all, there being no *resultant* pressure, but simply an equal pressure on back and front due to the depth below the surface, which would not be detected. In the case of the hose then the pressure in the jet is atmospheric simply, neither high nor low. We can also see this in another way; for, the pressure being equal in all directions, if the pressure in the jet were above the atmosphere, the outer portions would be thrown off radially, since the atmospheric pressure could not keep them together against the greater internal pressure.

Having now seen exactly what we mean by the pressure the question arises, How shall we measure it if required? Fig. 354 shows a pipe containing water. In the first case, suppose the water were still, the pressure could be measured by putting in a pipe of any shape as AB or CD with an open end; and the water

would rise to a height, say h, whence we should know that the pressure was wh.

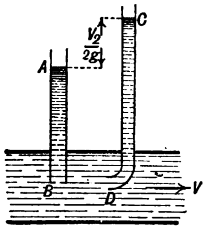


Fig. 354.

But now suppose the water is flowing with velocity V, then we find that in AB the water rises to A, while in CD it rises to C, and the difference of level of C and A will prove to be just $V^2/2g$ ft. The real pressure in the pipe is that shown by the tube AB, and the extra height of the column in CD is due to the fact of the still water in CD at its open end stopping the flow of the water which meets it, just as the hand held still in a running stream stops some water, and hence a pressure is felt. If there were no loss of head C would be then on the same level as the water surface in the reservoir.

The pipe AB is called a Piezometer, and we must always be careful to see that its end is quite parallel to the direction of flow.

Pipe of Varying Section.—We now consider what changes of velocity and pressure will take place in a pipe of which the sectional area is not constant. Fig. 355 shows a pipe, the sectional area of which varies *very gradually* (the reason for this we shall see a little later). In the pipe we place tubes AB, CD, A′B′, C′D′. Let

V = velocity at BD, V' = velocity at B′D′.
A = sectional area at BD, A' = sectional area at B′D′.

Then the most plainly evident thing perhaps that we know about the flow is that exactly as much water must flow past B in one second as flows past D in the same time, and hence

$$VA = V'A',$$

so that, if we know the velocity at any one point of a given pipe, we can at once determine it for any point whatever. Next, C and C′, being both on the level of

the water surface in the reservoir, are both on the same level. But A is $V^2/2g$ feet below C, while A′ is only $V'^2/2g$ feet below C′; whence it follows that A′ is $(V^2 - V'^2)/2g$ feet above A, and the pressure at B′, which would in still water be greater than at B by an amount wz—supposing B′ be z feet below B, *i.e* the head over B′ z ft. more than that over B—is now still further increased by the pressure due to $(V^2 - V'^2)/2g$ feet of head.

If the pipe be level then z vanishes, and we see that in a level pipe the pressure increases as the velocity

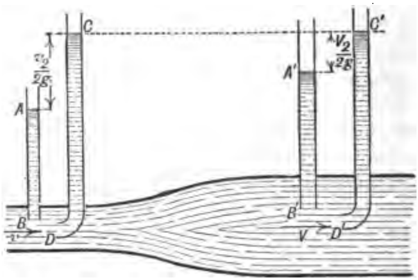


Fig. 355.

diminishes, *i.e.* as the sectional area increases. This result appears at first sight strange, and is often disputed by persons not properly acquainted with the elements of the subject. The objection is of course based on a misapprehension, being usually put something in this form: In the small part of the pipe the water must be more crowded together, and hence the pressure must be greater. The idea present here is plainly that of a crowd of people moving through a narrow passage between broader spaces, and where the analogy fails is in the fact that the velocity does not increase through the narrow part; those behind actively push, which we must remember a particle of water cannot do, and thus

prevent those in the narrow part from moving at a rapid rate; if in moving through such passages the rate of movement were made inversely proportional to the size of passage, then as each person came to the narrow part he would increase his speed, and thus relieve the pressure, and then on meeting those in front whose speed was again decreasing, he would press on them until his speed decreased again to the slower movement. It is the active pressing from behind, which living beings can exert, but inanimate matter cannot, that is the cause of the terrible results which often follow such movements.

Another remarkable result which follows from the preceding is, that there is a certain limiting speed beyond which water cannot be forced in a steady stream through a passage into the atmosphere.

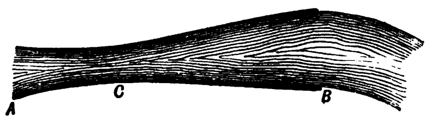


Fig. 356.

Let Fig. 356 represent a horizontal pipe, with an open end at B. Then at B the pressure is atmospheric. If

$$v = \text{velocity at C},$$
$$V = \text{velocity at B}.$$

Then

$$P_B = P_C + w\frac{v^2 - V^2}{2g}.$$

Now water cannot exert tension, or, in other words, P cannot be negative. If then we put $P_C = 0$, we have

$$v^2 = V^2 + 2g\frac{P_B}{w}.$$

But if

$$a = \text{sectional area at C},$$
$$A = \text{sectional area at B}.$$

$$V = \frac{a}{A} \cdot v.$$

Whence

$$v^2 = 2g\frac{\text{PB}}{w} \div \left(1 - \frac{a^2}{\text{A}^2}\right),$$

and if this velocity be exceeded, the pressure P_C must be negative. In this case what happens is that the water breaks away at C, and we get the kind of flow which we see when trying to pour water too quickly out of a bottle.

Discharge of Pipes.—One of the most important questions in hydraulics is to determine the necessary diameter of a pipe, such as a water main, to deliver a given quantity of water at a given point; or, which is the same thing, to find the discharge of a given pipe. ABCD

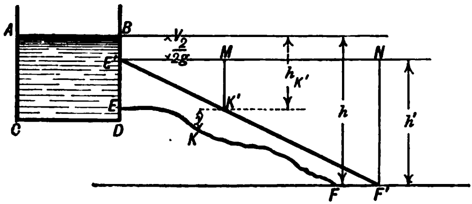


Fig. 357.

represents a reservoir, and EF a pipe discharging water at F (Fig. 357). The pipe is laid in the form shown, and it may in many cases be, at some points, below F. Let

l = length of pipe in ft.
d = diameter of pipe in ft.
h = head available in ft.

i.e. depth of F below AB.

V = velocity of delivery in f.s.

The head h has to give the water velocity and to overcome the surface friction.

The head wasted in the latter is

$$h' = 4f \cdot \frac{l}{d} \cdot \frac{V^2}{2g},$$

so that to produce the velocity we have left the head

$$h - h' = h - 4f \cdot \frac{l}{d} \cdot \frac{V^2}{2g},$$

$$\therefore \frac{V^2}{2g} = h - 4f \cdot \frac{l}{d} \frac{V^2}{2g},$$

or

$$\frac{V^2}{2g} = \frac{h}{1 + 4f \cdot \frac{l}{d}}.$$

[We neglect here the small waste of head on entering the pipe (see page 486)].

We thus have V, and then $Q = \pi/4 . d^2 V$ in c. ft. per second.

From the result of this work it appears that if any number of pipes of the same length and diameter discharge water at the same level, the velocity of discharge will be the same for all, and is thus independent of the part of the reservoir from which the pipe is led, or of the shape in which it is laid. Although the final result is the same, there are considerable differences in the circumstances of the flow at different points, and it is instructive to examine these differences.

We will compare EF with another pipe discharging on the same level, and will select for the comparison a straight pipe which leaves the reservoir at a depth $V^2/2g$ below the surface. Let E′F′ be this pipe.

First, we have at every point of each pipe the one velocity V, since this is the velocity at F and also at F′ by the preceding work.

The velocity at E′ is then V. But V is the velocity with which water would run out at E′ against the atmospheric pressure, since $h_{E'}$, or the head over E′, is $V^2/2g$ ft. It follows that the pressure in the pipe at E′ must be atmospheric; so there is atmospheric pressure at each end of E′F′. Moreover, we shall find that the pressure anywhere in E′F′ is atmospheric. For consider K′.

The head over K′ is $h_{K'}$ (Fig. 357). Of this an amount

$$4f\frac{E'K'}{d}\frac{V^2}{2g}$$

is wasted in surface friction between E and K′. Produce AB, and draw E′MN parallel to it. Also draw K′M, F′N vertical. Then at F′ the head wasted is h', and

$$h'+\frac{V^2}{2g}=h,$$

therefore, in Fig. 357,

$$h'=F'N,$$

$$\therefore\ F'N=4f\cdot\frac{l}{d}\cdot\frac{V^2}{2g}.$$

But

$$\frac{K'M}{F'N}=\frac{E'K'}{E'F'}=\frac{E'K'}{l},$$

$$\therefore\ K'M=\frac{E'K'}{l}\cdot 4f\cdot\frac{l}{d}\cdot\frac{V^2}{2g},$$

$$=4f\cdot\frac{E'K'}{d}\cdot\frac{V^2}{2g}.$$

So that K′M represents the head wasted between E′ and K′. Therefore

$$\text{Head available to produce velocity at } K'=h_{K'}-K'M=\frac{V^2}{2g}.$$

But this actually produces a velocity V, whence the pressure at K′ can only be atmospheric. K′ being any point, we have proved that at all points of E′F′ the pressure is atmospheric, and hence if a slit were cut in the top of the pipe, the water would not leak out, nor would air leak in.

More generally the pipe might be replaced by an open channel, having the same hydraulic mean depth as the pipe, so that h' was unaltered, and the water would flow steadily in this channel at the velocity V.

Such a channel or pipe is called the **Hydraulic**

Gradient, for all pipes of the given dimensions discharging on the same level.

For the slope of this channel we have, if it make an angle i with the horizontal,

$$\sin i = \frac{F'N}{E'F'} = \frac{4f \cdot \frac{l}{d} \cdot \frac{V^2}{2g}}{l},$$

$$= \frac{4f}{d} \cdot \frac{V^2}{2g},$$

and i is called the **Virtual Slope** of the pipes.

Pressure in a Water Main.—By means of the hydraulic gradient it is easy to determine the pressure at any point of a main such as EF (Fig. 357). Take a point K in EF corresponding to K′ in E′F′, *i.e.* at the same distance from E measured along the pipe as K′ is from E′. Then the head wasted before getting to K is the same as that wasted before reaching K′. But the head over K is more than that over K′ by y, where y is the vertical distance apart of K and K′; so there is at K a head $(V^2/2g+y)$ available. But this excess of head does not produce any increased velocity, since the velocity is V both at K and K′, whence it follows that the pressure resisting the flow at K must be greater than that at K′ by an amount equivalent to the excess y of head.

We have now a simple method of obtaining the pressure at any point. If K be the given point; take K′, the corresponding point as described above, and then if K be y ft. below K′, the pressure at K is wy lbs. per sq. ft. above the atmospheric pressure.

This is important in practice, especially in cases where K is above K′. So long as K is below K′, then if there be a leak at K water will leak out but air will not leak in. But if K be above K′, the pressure is less than that of the atmosphere, y being negative, and thus at a leak air would enter; also if there be no leak, still

the air which is always present in water would be released and collect at those points at which the pressure is least, and it would be no good simply fitting an air-cock, because when this was opened air would enter instead of that inside being driven out. In order then that air should not collect at such points and stop the flow, special means have to be used.

Siphons.—There is still another case to consider, viz. when y is not only negative but is more than 34 feet, *i.e.* the head due to atmospheric pressure. In this case the pressure at K would be negative, which we have seen is impossible, so that the flow would cease. This refers to the case of a siphon.

Taking (Fig. 358) K and K′, corresponding points,

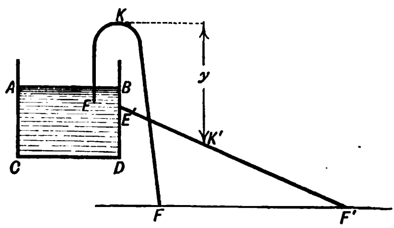


Fig. 358.

i.e. E′K′ = EK, the siphon will not flow if K is more than 34 ft. above K′. The usual statement in elementary hydrostatics is that K must not be more than 34 ft. above AB; this we see is not correct.

If K be more than 34 ft. above K′, but less than 34 ft. above AB, we get an intermittent flow; the water continually breaking away at K, then the leg fills again owing to the vacuum left at K, the flow commences again, then stops again, and so on.

Formula for Discharge.—When the length of a pipe is considerable compared with its length, the ratio

of head wasted to that usefully employed in giving velocity is large.

For example, in the formula

$$h' = 4f \cdot \frac{l}{d} \cdot \frac{V^2}{2g},$$

put $h' = V^2/2g$, then

$$4f \cdot \frac{l}{d} = 1,$$

and if $4f = .03$,

$$\frac{l}{d} = \frac{1}{.03} = 33,$$

so that in each length of $33d$ we waste head equal to that usefully employed. If then a 1 ft. diameter pipe be one mile long, the head wasted is $5280/33$, or 160 times that usefully employed; and we may say practically the whole head is wasted, and we could find V from the equation

$$h' \text{ or } 4f \cdot \frac{l}{d} \cdot \frac{V^2}{2g} = \text{total head available},$$

if this head were say 161 ft., then the error would consist in using 161 instead of 160, since 1 ft. only would be used in producing V. For this reason the discharge of a pipe is usually expressed in terms of h', the waste of head in it. We have

$$4f \cdot \frac{l}{d} \frac{V^2}{2g} = h',$$

$$\therefore V = \sqrt{\frac{g}{2f} \cdot \frac{h'}{l} \cdot d},$$

and

$$Q = AV = \frac{\pi}{4} \sqrt{\frac{g}{2f} \cdot \frac{h'}{l} \cdot d^5}$$

is the discharge in c. ft. per second, all dimensions being in feet.

Generally discharge is expressed in gallons per minute; for this we have

$$1 \text{ c. ft. per second} = 60 \times 62.5 \text{ lbs. per minute,}$$
$$= \frac{60 \times 62.5}{10} = 375 \text{ gallons per minute.}$$

Also, if d' be the diameter in inches, d in the preceding is $d'/12$, whence substituting, the discharge in gallons per minute is given by

$$G = 375Q,$$
$$= 375 \times \frac{\pi}{4} \sqrt{\frac{g}{2f} \cdot \frac{h}{l}} \cdot \left(\frac{d'}{12}\right)^{\frac{5}{2}},$$
$$= \frac{4.736}{\sqrt{f}} \sqrt{\frac{h'}{l}}\, d'^{\frac{5}{2}}.$$

We now drop the accent, recollecting d is to be in inches, and write

$$G = C \sqrt{\frac{h'}{l}}\, d^{\frac{5}{2}},$$

where $C = 4.736/\sqrt{f}$, and we take it as 30 for clean pipes 4 ins. diameter and above, to 24 for pipes 1 inch diameter.

Waste of Head—Other Causes.—In addition to surface friction, any sudden or partly sudden change of velocity either in magnitude or direction causes a waste of head. Some of these cases we will now consider.

Sudden Enlargement.—Fig. 359 shows a pipe, the diameter of which is suddenly enlarged. Let

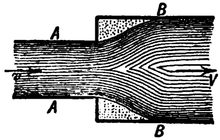


Fig. 359.

v = velocity in small part,
V = ,, large ,,
a = sectional area of small part,
A = ,, ,, large ,,

Then we have first

$$av = AV.$$

If the pipe were enlarged very gradually (page 475) the water would change its velocity quietly without any

eddying motions being set up in it. Actually, however, what happens is, that on opening into the large pipe the water breaks away in all directions; there is a sort of main stream flowing on, but the corners are full of broken water which is continually joining the main stream, and its place being supplied from the small stream entering. The water then flowing across the section BB has all kinds of cross motions, and head is wasted in producing these motions.

The effect is due to a stream moving with velocity v impinging on a larger stream moving at velocity V, the relative velocity or velocity of striking being $v - V$.

We will then first examine a somewhat simpler case.

Fig. 360 represents a bucket held stationary, the sectional area of the bucket being A; a stream of sectional area a enters the bucket at a velocity $v - V$ and strikes the bottom, being thus entirely stopped, and dropping down vertically when it pours out as more water enters. The head originally used to produce the velocity $v - V$ is thus wholly wasted, so the waste of head is $(v - V)^2/2g$, nothing but confused motions being finally produced by it.

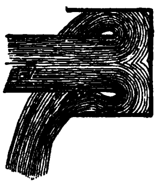


Fig. 360.

Next, let the whole system move on with velocity V, the bucket has now the velocity V and the stream a velocity v; the velocity of striking is not altered, and hence we infer that the amount of confused motion produced is the same, the water moving on also as a whole at the velocity V of the bucket. But the waste of head is due to the production of the confused motion, and we conclude that it will, therefore, as before, be $(v - V)^2/2g$.

The next step is obvious; we have only to replace the wooden bottom of the bucket by the water surface at BB, and we have the present case. We reason that this cannot affect the loss of head, it being immaterial

whether the surface struck be of water or of wood, and we say then finally

$$\text{Waste of head} = \frac{(v - V)^2}{2g}.$$

We have here a verification of the statement on page 469 regarding hydraulic resistances, for we will now express the loss by means of a coefficient of resistance.

We have two distinct velocities of flow, v and V, either of which may be selected, so that F, the coefficient, will have two values. We write then

$$\begin{aligned} F &= \frac{(v - V)^2}{2g} \div \frac{V^2}{2g}, \quad \text{or} \quad \frac{(v - V)^2}{2g} \div \frac{v^2}{2g}, \\ &= \left(\frac{v}{V} - 1\right)^2, \quad \text{or} \quad \left(1 - \frac{V}{v}\right)^2, \\ &= \left(\frac{A}{a} - 1\right)^2, \quad \text{or} \quad \left(1 - \frac{a}{A}\right)^2. \end{aligned}$$

or if we put $A = ma$,

$$F = (m - 1)^2, \quad \text{or} \quad \left(1 - \frac{1}{m}\right)^2.$$

In either case F is a constant, depending only on the nature of the source of the resistance, as stated in the general law. We must be careful to connect together the proper coefficient and velocity, to remember which take note that the larger coefficient goes with the smaller velocity, and *vice versâ*.

Sudden Contraction.—In Fig. 361 we have the reverse case to that just considered. The stream now contracts to CC and then expands again to BB, so that there are two distinct actions to consider.

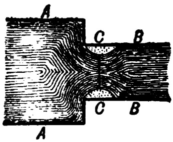


Fig. 361.

In the first of these little waste takes place, because little broken water is caused, the water moving quietly round the corners AA and being kept together

by the surface of the pipe. There will be a little loss, such as we have seen to occur in the case of a simple orifice.

But in the second action there will be considerable waste, this being a sudden expansion similar to that just considered, and consequently the waste being expressed in the manner just discussed. We neglect then the first part because the loss is so small, and we have

$$F=(m-1)^2,$$

where m is the ratio of a to the contracted section at C, and the coefficient refers to $v^2/2g$.

The value of m will vary with different ratios of A to a, and is believed to be approximately given by the empirical formula

$$m=\sqrt{2.618-1.618\frac{a^2}{A^2}}.$$

We see here the explanation of the increased waste of head caused by fitting a short pipe to an orifice (page 470), that being simply a case of the preceding, in which A is indefinitely large compared with a. We can calculate the amount thus. Let

$$A=\infty, \quad \therefore \frac{a}{A}=0.$$

$$\therefore m=\sqrt{2.618}=1.62,$$

and

$$F=(m-1)^2=(.62)^2=.3844.$$

The remaining part of the .505 is due to the friction of the corners of the orifice and of the pipe.

The preceding results and others which we cannot examine into are collected in the following table, the values of F being determined by experiment.

COEFFICIENTS OF HYDRAULIC RESISTANCE.

Nature of Obstacle.	Value of F.	Remarks.
Orifice (sharp-edged)	.06	
Square edged . } Entrance to a pipe	·5	
Sudden enlargement in ratio $m:1$.	$(m-1)^2$	Referred to smaller velocity.
Ordinary right-angled bend . . .	.14	Radius of bend $=3\times$ diameter of pipe.
Quick right-angled bend . . .	·3	Radius of bend = diameter of pipe.
Common cock partially closed. .	·75, 5·5, 31	Handle turned, 15°, 30°, 45° from fully open.
Surface friction of a pipe . . .	$4f.\frac{l}{d}$	For a clean iron pipe, diameter d'', $4f=.02\left(1+\frac{1}{d}\right)$ (Darcy's formula).
Right-angled knee .	1	

When water flows through a channel of varying section, containing several causes of resistance, the waste of head at each will be given as a multiple of $v^2/2g$, where v is the velocity past that particular obstacle. It is usual, for convenience, to express the

whole waste in terms of one selected velocity; suppose the velocity selected be V, then we have

$$\text{Waste at the obstacle} = Fv^2/2g \qquad \text{(F from the table)},$$
$$= F \cdot \frac{v^2}{V^2} \cdot \frac{V^2}{2g} = F' \frac{V^2}{2g},$$

where F' is a new coefficient derived from F by multiplication by v^2/V^2, or if a and A be the sectional areas at the parts considered, by multiplication equally by A^2/a^2. F' is called the coefficient of resistance *referred to the velocity* V. In this way values F', F'', etc., are obtained for all the obstacles, and then we have

$$\text{Total waste of head} = F' \frac{V^2}{2g} + F'' \frac{V^2}{2g} + \ldots,$$
$$= (F' + F'' + \ldots +) \frac{V^2}{2g},$$
$$= \Sigma F \cdot \frac{V^2}{2g},$$

ΣF being called the total coefficient referred to the velocity V.

Flow of Gases under Small Differences of Pressure.—When a gas flows, the density—*i.e.* w—varies as the pressure varies, and also varies with alterations of temperature. The flow then generally becomes a question of Thermodynamics.

If, however, the differences of pressure be small—that is, as is often the case in practice, such as are measured by a few inches of water—and no heat be supplied, the gas flows practically as a liquid having the same mean density, and we will examine this case. Let

T = absolute temperature of gas,

then $T = 461 + t$, where t is the Fahrenheit temperature,

V = volume of 1 lb. in c. ft.,

w = weight of 1 c. ft. in lbs. $= \frac{1}{V}$,

ΔP = the small difference of pressure producing the flow in lbs. per sq. ft.

Then the head equivalent to ΔP is $\Delta P/w$ or $V\Delta P$ feet, and the velocity of discharge u will be given by

$$u^2 = 2gV.\Delta P.$$

ΔP is measured by a siphon gauge in inches of water. Let

i = the difference of pressure in inches of water,

then

$$1 \text{ inch of water} = \frac{62.5}{12} = 5.2 \text{ lbs. per sq. ft.}$$

$$\therefore \Delta P = 5.2i.$$

The mean pressure will be known, say P; then V is given by the formula

$$PV = cT.$$

Whence

$$u = \sqrt{2g\frac{cT}{P} \cdot 5.2i}.$$

Taking now the definite case of air, and taking P to be the ordinary atmospheric pressure, the values are

$$c = 53.2, \quad P = 2116.$$

Whence

$$V = \frac{T}{40},$$

and

$$u = 2.89\sqrt{Ti}.$$

The volume of gas discharged per second per sq. ft. of effective, *i.e.* contracted, area of orifice is u c. ft., and its weight is therefore given by

$$W = uw = \frac{u}{V} = \frac{2.89\sqrt{Ti}}{\frac{T}{40}},$$

$$= 115.6\sqrt{\frac{i}{T}},$$

so that for a given value of i the weight discharged decreases as the temperature rises.

The head producing flow is VΔP, or substituting becomes

$$h=\frac{\mathrm{T}i}{7\cdot 7},$$

which increases with the temperature ; this head, it must be remembered, is feet of the gas, not feet of water.

Thus, as T increases, the head due to a given difference of pressure increases, so that the velocity and volume of discharge increase ; but the density decreases faster, so that the weight discharged is less.

If the flow take place through an orifice, coefficients of resistance and contraction must be allowed, and they may be taken as having the same values as for water. It is of course evident that they are much more liable to variation from small causes. In flow through a pipe, the head wasted in overcoming surface friction is given by

$$h'=4f\cdot\frac{l}{d}\cdot\frac{u^2}{2g},$$

the work on page 472 applying to all cases, and, by the law of resistance on page 471, f has the same value.

The discharge in c. ft. per second of a given pipe will accordingly be the same as on page 483, viz.

$$\mathrm{Q}=\frac{\pi}{4}\sqrt{\frac{g}{2f}}\sqrt{\frac{h'}{l}}\cdot d^{\frac{5}{2}},$$

$$=40\sqrt{\frac{h'}{l}}\cdot d^{\frac{5}{2}}\quad(d\text{ in ft.}),$$

taking a mean value for f. The weight discharge must contain T, so cannot be expressed as for water.

EXAMPLES.

1. A circular tank, 20 ft. diameter, is constructed of $\frac{1}{2}$ inch iron plates. Find the greatest depth of water in it, the stress not being more than 4000 lbs. per sq. inch of the solid metal.

Ans. If h be the depth, the tank at the bottom is exposed to an internal bursting pressure wh lbs. per sq. ft. Whence $h=38.4$ ft.

2. The tank above is 120 ft. above the ground. Find the necessary thickness for a 1 inch copper service pipe on the ground level. Allowing 2000 lbs. per sq. inch. *Ans.* $\frac{1}{68}$ inch.

3. The discharge from a 2 inch circular orifice in the side of a tank 8 feet below the water level just clears the edge of a 100 gallon tank, distant 11 feet horizontally and 4 feet vertically; also the tank is half filled in 26 seconds. Find the velocity of discharge and the coefficients of velocity and contraction.

Ans. 22 f.s.; .97; .623.

4. Calculate the time required to sink an iron tank 30 feet long, 20 feet broad, and 9 feet deep, the water entering through an orifice 3 ins. diameter in the thin bottom, supposing the tank when empty to float with 5 feet out of water.

Ans. $1\frac{3}{4}$ hr. nearly.

5. Find the result of the preceding when a pipe $4\frac{1}{2}$ ins. long is fitted square to the orifice inside the tank.

Ans. 1 hr. 18 min.

6. The barometer stands at 30 ins., and the vacuum gauge on a condenser shows 26 ins. of vacuum. The injection orifice is 10 ins. diameter, 8 feet below the sea-level, and the pipe connecting it to the condenser is 5 feet long. Find the quantity of water entering per second. $4f=.021$. *Ans.* 6.55 c. ft.

7. A cylindrical boiler is 12 feet diameter, and the water level is at $\frac{3}{4}$ the diameter from the bottom; the steam pressure is 120 lbs. by gauge. Find the velocity with which water would flow out—1st, through a hole in the bottom into the stokehold; 2d, through a 6 inch pipe 8 feet long into the sea, the bottom of the boiler being 15 feet below the sea-level. *Ans.* 130 f.s.; 92.3 f.s.

8. A 4 inch pipe, running full, delivers 120 gallons of water per minute. Find the hydraulic gradient or virtual slope.

Ans. $\frac{1}{64}$.

9. Water issues from the nozzle of a fire hydrant 1 inch diameter with a velocity sufficient to project the jet to a height of 100 feet. Determine the pressure in the hose near the nozzle, the internal diameter being 3 inches. Neglect the effect of friction.

Ans. 43 lbs. per sq. inch above atmospheric.

10. Find the velocity of flow of water in a rectangular canal 30 feet wide by 5 deep, sloping 18 inches per mile. Coefficient of friction .012.

Ans. The head wasted in friction per mile is $1\frac{1}{2}$ ft., and the hydraulic mean depth is $3\frac{3}{4}$ ft., whence $V = 2.4$ f.s.

11. A pipe 5 ins. diameter delivers a certain quantity of water per minute with a loss of head of 4 ft. Determine the loss of head if the same quantity were delivered through a pipe 4 ins. diameter, assuming the same coefficient of friction.

Ans. 12.2 ft.

12. The diameter of a screw propeller is 15 feet, pitch 18 feet; neglecting slip, find the horse power wasted in overcoming the friction of 1 sq. ft. of blade at the tip, at 100 revolutions. Coefficient .005. *Ans.* 10.8.

13. A 1 inch circular hole in the side of a tank is fitted with an expanding nozzle 2 ins. diameter at its open end. Find the greatest depth of water over the hole for which steady flow is possible, neglecting all friction or contraction.

Ans. 2 ft. 4 ins.

14. A siphon 4 ins. diameter, with its end 1 foot below the water level in the source, discharges water on a level 6 feet lower. The total length is 80 feet, and its highest point is 36 feet from the entrance end. Find the discharge, and the greatest height possible for continuous flow.

Ans. 240 galls. per min.; 30.8 ft. above water level.

15. A 4 inch pipe delivers 100 gallons per minute into a 6 inch pipe, the axis of the two lying in one horizontal line. The pressure in the 4 inch pipe is atmospheric. Find the waste of head at the entrance, and the pressure in the larger pipe.

Ans. Waste of head 3 ft. 6 ins. Had no head been wasted the pressure would be 18.6 lbs. per sq. inch; the waste of head is equivalent to a loss of pressure $1\frac{1}{2}$ lbs., hence the pressure is 17.1 lbs. per sq. inch.

16. Obtain the second result of question 7, when a cock in the pipe is half closed, and allowing for two ordinary bends.

Ans. 23 f.s.

17. 1000 c. ft. of water per minute are pumped through the surface condenser of a marine engine, and discharged into the sea through an orifice 27 ins. diameter. The I.H.P. of the pumping engine is 25. Assuming the mechanical efficiency of the engine and pump combined to be .5, estimate the coefficient of resistance referred to the velocity of discharge.

Ans. Work done on water per minute equals 412,500 foot-lbs., which would lift the 1000 c. ft. through 6.6 ft. The total head then to produce V and overcome friction is 6.6 ft., whence $F = 21.5$.

18. The difference of pressure between the two ends of a pipe 6 ft. long, 6 ins. diameter, is 3 ins. of water. Find the speed with which air at atmospheric pressure would flow through—1st, at 60 F.; 2d, at 600° F. *Ans.* 107 f.s.; 140 f.s.

19. The air pressure in a stokehold is 2 ins. of water, find the quantity of air which would be discharged per minute through a hole 2 ins. diameter in the casing. Temperature 90° F.

Ans. 53 lbs.

NOTE.—To avoid misapprehension, it may be added that cross motions, such as are shown in Fig. 348, page 465, and referred to on page 469 and elsewhere, would not be possible in the absence of friction and discontinuity. For an explanation of the way in which energy is dissipated in fluids by the formation of eddies which are subsequently extinguished by fluid friction, advanced students are referred to chapter xx. of the larger treatise.

CHAPTER XXIV

HYDRAULIC MACHINES

A HEAD of water may be utilised by employing a machine driven by the water, and in the present chapter we shall consider some of the simpler types of these machines.

Weight Machines.—The most simple class of

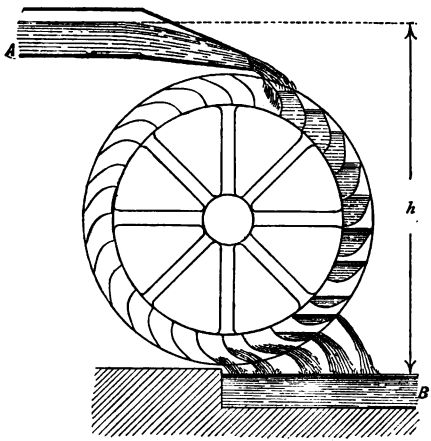


Fig. 362.

machine which can be driven by water from an elevated reservoir is that in which the water during its descent

rests on the buckets or vanes of a wheel, so that its weight becomes the effort driving the wheel.

Overshot Wheel.—In Fig. 362 we have one example of this class, the common overshot wheel. The water from the reservoir A pours into the buckets of the wheel, and by its weight turns the wheel, the buckets emptying into the tail race B.

The total head available is h, and in descending this distance the energy exerted by gravity on a weight W of water is Wh. Let

$$G = \text{delivery of stream in gallons per minute.}$$

Then

$$\text{Energy exerted per minute by gravity} = 10\ Gh \text{ ft.-lbs.}$$

The whole of this energy cannot be utilised. For, in the first place, the water runs on to the wheel with a velocity v; on striking the wheel vanes a pressure is created by the sudden change of velocity to V, the speed of the vanes or buckets (compare page 486), so that some of the head $v^2/2g$ originally used in producing v is usefully employed in creating a pressure helping to drive the wheel, but we know there is a waste of head $(v - V)^2/2g$ at least. The value of V is limited, because if it become large the water will be thrown out of the buckets by centrifugal action, hence V does not exceed about 5 f.s.

For reasons which we cannot in the present book enter into v should be about twice V, so that v is 10 f.s. Hence

$$\text{Head wasted} = (10 - 5)^2/2g,$$
$$= \frac{25}{64.4}.$$

Again, the water being in the buckets has the velocity V, and thus a portion of the head is wasted in this way, since the velocity is of no use to us. For this loss we have

$$\text{Head wasted} = V^2/2g.$$

We may add this to the preceding, and we then obtain

$$\text{Head wasted in shock and in giving velocity V} = \frac{(v-V)^2}{2g} + \frac{V^2}{2g},$$

which if $V = v/2$ becomes

$$\tfrac{1}{2} \cdot \frac{v^2}{2g} = \frac{50}{64.4} \text{ ft.},$$

if v be 10 f.s.

There is still another loss caused by the buckets emptying before reaching the bottom of the fall, and by spilling of water. This loss depends on practical con-

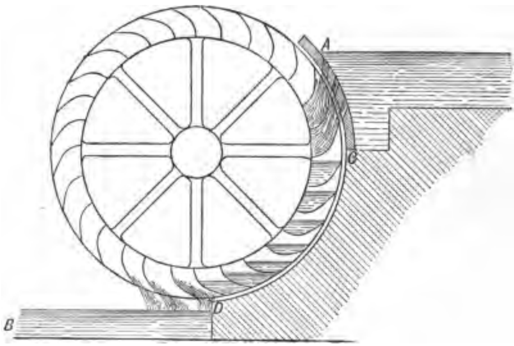


Fig. 363.

siderations as to shape of buckets, and hence we cannot express it by a formula.

We see that the losses or wastes contain some of constant value, and hence their relative effect is greater in small falls. For this reason these wheels are not used for values of h less than 10 feet. There is a limit on the other side, because for a very high fall the wheel becomes of very great diameter and too cumbrous. The limit lies probably between 60 and 70 feet. The efficiency obtained in practice varies between 65 and 75 per cent.

Breast Wheel.—In order to avoid the loss by

spilling of water from the buckets, a wheel is used which moves in a masonry channel; the buckets are replaced by vanes which fit this channel, but not so closely as to touch the sides; so there is a leakage past the vanes, but this does not cause so much loss as the buckets do.

Fig. 363 shows the construction, CD showing the breast or channel. It will be seen that the diameter of the wheel is necessarily greater than the fall, so that these wheels cannot be used for falls exceeding about 50 ft. The average efficiency is about .75. The water enters the wheel through guide blades at A, which are arranged so as to prevent as much as possible any shock as the water enters the buckets, and the waste of head which accompanies such shocks.

Pressure Machines.—In the preceding machines the water has been open to the atmosphere, and each portion contained in a bucket or between two successive vanes has acted simply by its weight. In the class of machine we are now about to consider the water is confined within a pipe which is led from the source to the working cylinder of the machine, and the pressure due to the head moves the piston; the water is then discharged just as steam is exhausted from a steam engine.

Hydraulic or Bramah Press.—The simplest machine of this class is one in which there is practically no motion, but the effort is very much magnified. The head is in this case produced not by an elevated reservoir, but by loading a piston CD with P lbs. (Fig. 364). CD fits a cylinder, area A_{CD}, which is connected by a pipe with the larger cylinder, area A_{AB}, in which fits the working piston AB. AB presses against the body which is to be pressed, and exerts on it a total pressure R, the reaction R of

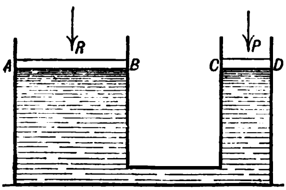


Fig. 364.

the body being the resistance to the motion of AB, as in former examples.

Assuming for simplicity that AB and CD are on the same level, we have, since there is no motion — the body pressed being supposed now to be compressed to the full amount—equal intensity of pressure at AB and CD. Hence

$$\text{Total pressure on AB} : \text{Total pressure on CD} = A_{AB} : A_{CD},$$

or

$$R : P = A_{AB} : A_{CD},$$

so that by making the ratio of areas very large, a small

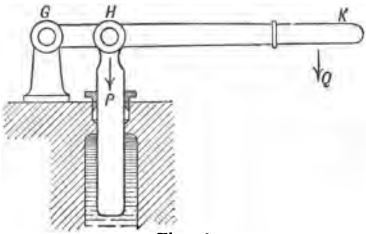


Fig. 365.

load or effort applied to CD can exert a great pressure at AB.

In the actual press a plunger takes the place of the piston CD, as in Fig. 365, and the effort P is applied to the top of the plunger by a hand lever GHK, to the end K of which a force Q is applied. A_{CD} is now the sectional area of the plunger.

Accumulators.—In some cases natural sources of head are available to work pressure machines, but in the majority of cases the head is produced artificially, as in the case we have been considering, or as in the hydraulic engines used on board ship, in which a head is first created by means of a steam engine. In one or two cases the engine which creates the head has actually

pumped water up to a tank on an elevation, but this is unnecessary, since the same effect can be produced by the use of an accumulator. This apparatus consists simply of a cylinder in which a piston or plunger works, this piston being loaded with heavy weights.

Fig. 366 shows the arrangement, the weights being shown resting on the piston rod head; actually they are not applied in this way, but the principle is unaffected. Let

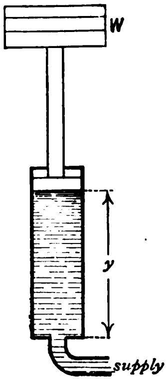


Fig. 366.

y = greatest height of piston in feet,
W = load in lbs.

We can always neglect the pressure due to the actual head h, this being so small compared with the effect of W, which ranges up to 1500 lbs. per sq. inch or even more. Then the accumulator contains a store of energy Wy foot-lbs., which can be given out by allowing the loaded piston to descend.

The store of energy in the accumulator would not drive the machines for any great time, but during the working it is being continually supplied by the steam engine. For example, in riveting, the engine supplying the accumulator continues working until the piston is right up, this shuts off steam; now a rivet is compressed, the piston falls, opening the steam valve, and the engine starts working again, and so on.

Taking a pressure of 1500 lbs. per sq. inch. which is used for riveting, this would require a head of

$$\frac{1500 \times 144}{62.5} = 3456 \text{ ft.},$$

which it would be practically impossible to obtain.

Generally, we have the head equivalent to W given by

$$h = \frac{W}{A} \div w,$$

A being the piston area, and w the weight of a c. ft. of water.

Pressure Machine in Steady Motion.—When a pressure machine is moving steadily at a constant speed, the pressure in the accumulator has to overcome the useful resistance to the moving piston, the friction of the moving parts, and the hydraulic resistances to the motion of the water. These last are of great importance, and we must see what effect they produce.

Fig. 367 shows a cylinder C with a piston B. The cylinder is supplied by a pipe A from the accumulator. Let

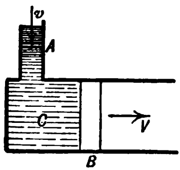


Fig. 367.

P_o = pressure in accumulator,
P = pressure resisting motion of piston,
= useful resistance + solid friction,
V = velocity of piston,
v = velocity of water in pipe.

There is now between the accumulator and the working piston a fall of pressure $P_o - P$, so that head equivalent to this fall of pressure must have been wasted in overcoming hydraulic resistances. Hence

$$\frac{P_o - P}{w} = \text{head wasted in resistances,}$$

$$= F \cdot \frac{v^2}{2g},$$

where all the separate wastes are referred to the one velocity v (page 489). We choose v in the first instance as the most natural velocity to refer to, but it is more convenient to use V, that being the most easily determined in most practical cases. Let

D = diameter of cylinder,
d = diameter of pipe.

Then

$$V : v = d^2 : D^2,$$

$$\therefore V = v\frac{d^2}{D^2},$$

and

$$\frac{V^2}{2g} = \frac{v^2}{2g} \cdot \frac{d^4}{D^4},$$

Hence the relation between P_o, p, and v becomes now

$$\frac{P_o - P}{w} = F \cdot \frac{D^4}{d^4} \cdot \frac{V^2}{2g} = F_o\frac{V^2}{2g},$$

where F_0 is the coefficient of hydraulic resistances referred to the velocity V.

From the formula last obtained we see that for a given value of P, V has a certain definite value, *i.e.* that there is only one particular speed at which the machine can work steadily, and this is called the speed of steady motion. When first started the speed will increase up to this, but will not go beyond so long as P and F remain unaltered. Now this property is not shared by ordinary machines; in all ordinary cases, if the effort be more than sufficient to balance the resistance, the speed will go on increasing indefinitely, *e.g.* the racing of an engine, and appliances as governors or brakes must be fitted to prevent this. But a hydraulic machine cannot exceed the speed just found, and so *contains automatic brakes within itself.*

Moreover the speed can be adjusted to any extent we please by altering F, which can be easily effected by means of a cock placed in the supply pipe (see table, page 488).

Examples of Pressure Machines.—The working of these machines cannot be fully studied without considering the forces necessary to produce the accelerations of the moving water which necessarily accompany those of the piston, but this is beyond our limits; and so we can only briefly mention a few examples of this type of machine.

1. Direct acting lifts. Here a platform rests on the end of a plunger or ram which is forced out of cylinder by water supplied from a tank. The tank may be used in this instance instead of an accumulator, as the head required is not very great; and besides, since the head decreases as the plunger rises, the velocity is not increased so much as the lift rises, and it can be more easily stopped than it could be were the head constant.

The simple figure (Fig. 368) shows all that is necessary for our purposes. A is the platform moving between guides, B is the ram, C the hydraulic cylinder, supplied by the pipe D from the tank at a height h. The ram does not fit the cylinder, so that the water pressure P is not exerted on an area $\pi/4 \, . \, D^2$ where D is the diameter of cylinder, but $\pi/4 \, . \, D'^2$, D' being the diameter of the ram; the diameter of the cylinder is of no consequence.

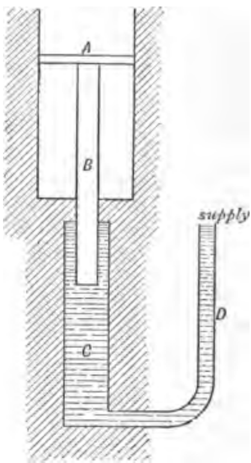


Fig. 368.

We have now an actual head h, so that for P_0 we write wh in the preceding formulæ, and hence determine V_0, the speed of steady motion. The calculation of the effect of the variation of h as the ram rises is beyond our present powers.

2. In a direct acting lift, the length of ram and working cylinder must be more than equal to the total lift, so much space is occupied. To avoid this the platform is in many cases lifted by blocks and tackle, used in the reverse way to that in which we originally considered them. In that case we wished to magnify

the effort, but in the present case we can easily obtain any required effort simply by increasing the area of the ram, and we then apply the tackle to magnify the velocity. The working cylinder can be placed in any position, as is most convenient for space; thus for working a derrick on board ship, the cylinder has at various times been bolted to the deck, then to the mast, and in some cases to the jib of the derrick.

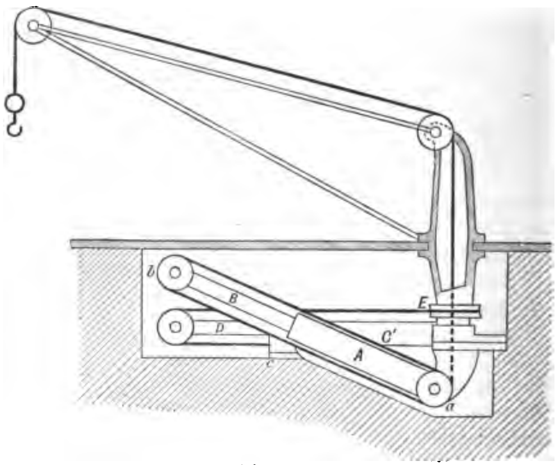


Fig. 369.

Fig. 369 shows the application of hydraulic cylinders to a crane. A is the cylinder placed for convenience in a sloping position, B is the ram, the blocks *a*, *b* are fastened to the cylinder framing and the end of the ram respectively, and they may contain one, two, or more sheaves as required; the chain, which in the direct manner of using blocks would be acted on by the effort, is now led up through the hollow crane post and over fixed pulleys to the weight to be lifted. If now there

were only one sheave in the movable block on B, the thrust of the ram would be 2W when lifting a weight W, and the motion of the ram would be only half that of the weight. For two sheaves at B the motion would be magnified four times, and so on exactly as on page 114.

The crane is slewed by hydraulic cylinders, arranged as shown in plan by Fig. 369, and in elevation in Fig. 370. C and C′ are equal cylinders, D and D′ the rams. A chain is fastened below C, passes over a pulley on the end of D, then round the sheave E on the crane post, over the pulley on the end of D′, and is then fastened to a similar point below C′. Evidently if D be thrust out the crane is rotated clockwise, and D′ is pulled in, and *vice versâ* if D′ be thrust out. There is no energy expended other than that required to overcome the friction,

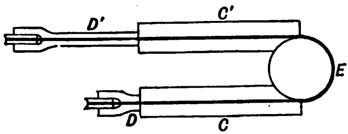


Fig. 370.

if the pressure on the water forced out of D′ be utilised, but if not then there is just as much energy used as if the ram C had been thrust out against the full load it could overcome.

Differential Rams.—The last point we have mentioned is of great importance, and requires examination.

In all cases we have a constant head, h say, producing the pressure P_0. If then A be the area of the ram, a load P_0A can be lifted at a very slow speed; if the load be less than P_0A, the speed must be controlled by means of the frictional resistance, but in all cases the effort is P_0A, and during a lift y energy P_0Ay must be exerted. If the load be small the greater part of this energy is necessarily wasted in overcoming the increased friction which must be applied to keep the motion steady,

and it becomes of importance if possible to avoid this waste.

We cannot prevent the waste, as in a steam engine, by cutting off the supply, since the water would not expand as steam does, but it can be decreased by the use of the differential ram.

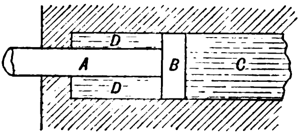


Fig. 371.

In Fig. 371 A the ram has a piston end B which fits the hydraulic cylinder C. The spaces C and D can be put into communication with either the supply pipe or the exhaust pipe. Let

A_C = area of cylinder,
A_A = area of ram,
s = stroke.

Then

$A_C - A_A$ = area of piston exposed to the water pressure in D.

When a heavy weight is lifted, C is open to supply and D to exhaust, and energy $P_0 A_C s$ is used per stroke. But when a smaller weight is lifted both C and D are connected to the supply, so that the ram moves out under a pressure,

$$P_0 A_C - P_0(A_C - A_A) = P_0 A_A,$$

and less energy is used per stroke; or, looking at it another way, a less quantity of water is used from the supply pipe, the quantity $A_C s$ enters C, but a quantity $(A_C - A_A)s$ is forced from D into the pipe, so the quantity used is only $A_A s$ instead of $A_C s$. This latter consideration shows also the gain of energy, for to force the water to the elevation h, or into the accumulator against the pressure P_0 or wh, requires the doing by the engine of an amount of work wh ft.-lbs. for every lb. of water supplied.

This device is also used in the Differential Accumulator, Fig. 372.

The central spindle is in two lengths of different diameters, and the cylinder slides on it, the water entering through the dotted channel through the centre of the spindle. The effective area is the difference of the areas of the two parts of the spindle, and thus the latter can be made stout, without requiring the weights W to be very large to produce the required pressure.

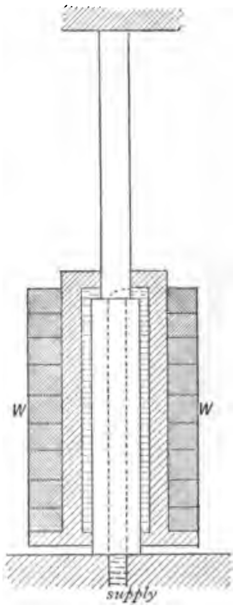


Fig. 372.

3. In the third type of pressure machine, the water drives an engine almost identical in its arrangement with a steam engine; no fresh principle is involved, and want of space forbids our entering into details.

Hydraulic Brake.—The hydraulic cylinder is often used simply as a brake.

Fig. 373 shows a common form; the rod A is connected to the body whose motion is to be controlled, and as the rod moves the water is driven by the piston B through the pipe C from one end of

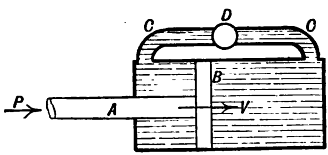


Fig. 373.

the cylinder to the other. In C is a cock D, by means

of which the value of F is made anything we please; there is now no useful resistance.

If the body is always to be brought up in about the same space, and is subject to the same forces, as a gun fired with its ordinary charges, then F will not require to be varied, and the pipe and cock may be dispensed with, their place being taken by a hole or holes in the piston itself, the resistance to flow through the orifices supplying the necessary retarding force.

Pumps.—If a pressure machine be worked backward, the water passes from the exhaust pipe into the cylinder, and is then forced back into the accumulator or reservoir through the supply pipe. But a machine which does this we call a Pump, so that a pump is simply a reversed pressure engine or motor, and the theory of its action is similar to that of the engine.

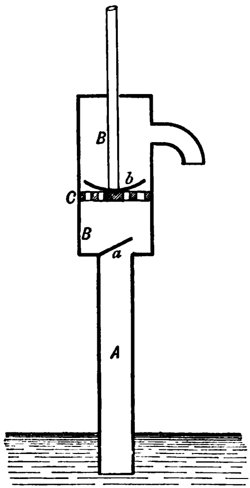


Fig. 374.

This kind of pump can be divided into two classes, viz. Lift or Bucket Piston Pumps, and Force, Plunger, or solid Piston Pumps.

Lift Pump.

Fig. 374 shows the construction of a lift pump. A is the suction pipe dipping in the water, its top can be closed by the valve *a*, B is the cylinder, C the piston or bucket, perforated to admit of the passage of fluid, but the holes can be closed by the indiarubber or leather valve *b*. The first few strokes remove the air from and thus lessen the pressure in A, so that the water rises in it to a height *h*, given

by $(P_0 - P)/w$, where P_0 is the atmospheric pressure on the water surface, and P is the pressure of the air under C; this height will not exceed 25 feet, since a perfect vacuum cannot be obtained Now as C descends the water passes through it, but cannot re-enter A since the valve *a* will shut (in Fig. 374 they are both shown open for clearness, but this cannot be during the working), then as C rises *b* closes and the water above C is lifted to the spout whence it flows, while the reduction of pressure under C allows the valve *a* to open and water to enter the lower part of B, following C up, so long as C does not rise more than about 25 ft. above the surface, as stated above.

If a somewhat higher lift be required it can be obtained by fitting a valve or valves to the top of the pump, as shown at *c c* in Fig. 375. The bucket is rising, *b* is shut and *a* open, and *c c* are also open to allow the water to pass through them and by the pipe D to the required place. When the bucket descends the valves *c c* close, so the water cannot follow the plunger down, and prevent the valve *b* lifting, as it would do if *c c* were not fitted. *c c* are called the Head Valves, and *b* the Foot Valve.

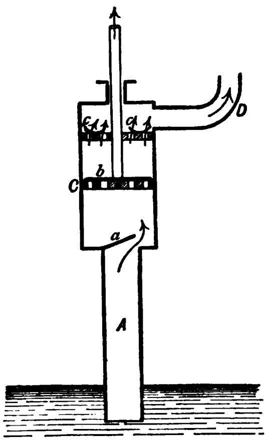


Fig. 375.

Force Pump.—When a great lift is desired then the bucket valves do not work well, and it is better to have a solid piston or plunger.

Fig. 376 shows the arrangement. The delivery pipe D now leads from the same end as the suction pipe, and

a valve *b* is fitted to its entrance, known as the "heading" or "delivery" valve. During the up-stroke *a* opens, and *b*, which opens towards the pipe, shuts, so the water rises under C; then during the down-stroke *a* shuts and the water is forced through *b* into the delivery pipe. There is no limit except the strength of the parts to the height of delivery in this case.

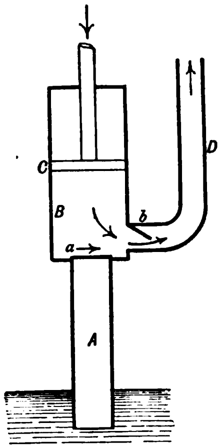


Fig. 376.

The lift pump is necessarily single acting, but the force pump is often double acting, as in Fig. 377.

The cylinder is shown horizontal, a branch from the suction leading to each end, and the same for the delivery. There are two sets of foot valves and also of delivery valves, so that while one end is in connection with the suction the other is dis-

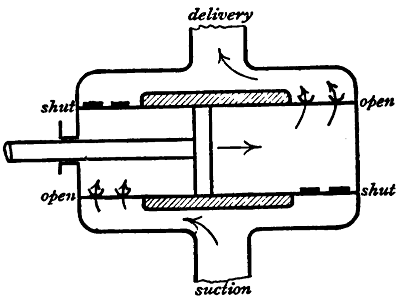


Fig. 377.

charging and *vice versâ*, and we thus get the discharge of two pumps while occupying little more space than one.

In all these cases the quantity of water actually delivered is less than As c. ft. per stroke,—A = area of piston, s = stroke,—because of the leakage back past the valves, the loss or slip varies from 8 or 10 per cent up to 25 or even more; it is least in vertical pumps, in which the bucket is used, and greatest, so far as the types we have considered are concerned, in horizontal plunger, or solid piston, pumps.

EXAMPLES.

1. A breast wheel 50 feet diameter receives 5000 gallons per minute, total fall 48 feet. Find the revolutions in order that the velocity of periphery may be 5 f.s., and if the H.P. be 52, find the efficiency. *Ans.* 1.91; .715.

2. If the preceding were replaced by an overshot wheel in which the buckets commenced to empty themselves at 5 ft. from the bottom, what H.P. would be lost from this cause? *Ans.* 7.6.

3. The ram of an accumulator is 8 ins. diameter, and is loaded with 20 tons. To what head is the pressure equivalent? If the stroke be 8 feet, how much energy is accumulated when the ram is right up? *Ans.* 2050 ft., 160 ft.-tons.

4. The diameter of the large part of the spindle of a differential accumulator is 8 ins. Find the diameter of the small part so that a load of 12 tons may be equivalent to a head of 1800 feet. *Ans.* $10\frac{3}{8}$ ins.

5. The accumulator in the last question is connected to a hydraulic cylinder 8 ins. diameter by a 2-inch pipe 50 feet long. Find the speed of steady motion when the total resistance of load and solid friction is $1\frac{1}{4}$ ton. *Ans.* 6.9 f.s.

6. A load of 5 tons is to be lifted 20 feet by a hydraulic crane. The water pressure is 700 lbs. per square inch, and the total efficiency one-half. Find the necessary volume of the ram. *Ans.* $4\frac{4}{5}$ c. ft.

7. If the ram in (6) be differential, with area of rod half that of piston, find the saving of energy when lifting $2\frac{1}{2}$ tons, assuming the same efficiency. By how much would the coefficient of resistance referred to the velocity of lifting require to be increased if the ram were not differential? *Ans.* 100 ft.-tons per lift; trebled.

8. A lift is required to raise 2 tons at 5 f.s.; the diameter of the supply pipe is 4 ins., length 200 feet. Find the head required, the ram being 18 ins. diameter, and its stroke half the rise of the lift. *Ans.* 680 ft.

9. A pump piston is 6 ins. diameter, 9 ins. stroke; it pumps into a boiler against 135 lbs. pressure, through a 4-inch pipe, 30 feet long, with three bends. Find the necessary steam pressure on a 10-inch piston to run at 40 revolutions per minute.

Ans. 52 lbs. per square inch.

10. Prove that the tension of the rod in a common pump is A*wh*, where A is the area of piston, *w* the weight of a c. ft. of water, and *h* the height of the water-level in the pumps, whether above or below the bucket, above the surface of the water outside.

11. The length of the cylinder of a hydraulic brake is 3 ft., diameter 12 ins., diameter of pipe connecting the ends 2 ins., length 3 ft. 8 ins. Find the force necessary to move the piston at 4 f.s. when the cock is wide open, and half shut, respectively.

Ans. 15.3 tons; 54.3 tons.

12. In (10) find the delivery, allowing 20 per cent slip.

Ans. 59 galls. per min.

THE END

Printed by R. & R. Clark, *Edinburgh.*